On the Maximum Number of Hamiltonian Paths in Tournaments

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Abstract

By using the probabilistic method, we show that the maximum number of directed Hamiltonian paths in a complete directed graph with n vertices is at least $(e - o(1))\frac{n!}{2^{n-1}}$.

1 Introduction

A *tournament* T is an oriented complete graph. A *Hamiltonian path in* T is a spanning directed path in it. Let $P(T)$ denote the number of Hamiltonian paths in T. For $n \geq 2$, define $P(n) = \max P(T)$, where T ranges over all tournaments T on n vertices. More than 50 years ago, Szele [5] showed that

$$P(n) \geq \frac{n!}{2^{n-1}} \tag{1}$$

His proof is considered to be the first application of the probabilistic method in combinatorics, and is thus mentioned in the beginning of most books and survey-articles on this method.

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The proof of (1) is very simple. Given a complete graph on n vertices, orient every edge independently with equal probability for each of the possible 2 orientations. Given a permutation of the vertices, it is clear that the probability that the permutation forms a Hamiltonian path is $\frac{1}{2^{n-1}}$. Hence, the expected number of Hamiltonian paths, which cannot exceed $P(n)$, is $\frac{n!}{2^{n-1}}$.

In this note we improve this bound by a factor of $e - o(1)$, as stated in the following theorem.

Theorem 1

$$P(n) \geq (e - o(1))\frac{n!}{2^{n-1}},$$

where the $o(1)$ term tends to 0 as n tends to infinity.

Like the proof of (1), our result is based on calculating the expected number of Hamiltonian paths in randomly generated tournaments. However, in order to increase the expected number of Hamiltonian paths, we use a scheme in which the edges along triangles are oriented together.

Let $\mathcal{T}$ be a collection of t pairwise edge-disjoint triangles in the complete graph K_n, and let T_n be the random tournament obtained by orienting each member (i, j, k) of $\mathcal{T}$, randomly and independently, either $(i, j), (j, k), (k, i)$ or $(i, k), (k, j), (j, i)$ with equal probability, and by orienting each edge not in any member of $\mathcal{T}$ randomly.

For a permutation $s = (i_1, i_2, \ldots, i_n)$ of the vertices of T_n, and for a triangle $R = (i, j, k)$ in $\mathcal{T}$, we say that R is present in s if the three vertices of R appear in three consecutive places of s. Let $X(s)$ denote the number of triangles in $\mathcal{T}$ which are present in s. The crucial point is the observation that the probability that s is a directed Hamiltonian path in our random tournament T_n is precisely $\frac{2^{X(s)}}{2^{n-1}}$. That is, if $X(s) = 0$ then the probability that s is a directed Hamiltonian path is precisely what it is in the usual random tournament, whereas in any other case it is strictly bigger. Therefore, the expected number of such paths in T_n is

$$\sum_{s \in S_n} \frac{2^{X(s)}}{2^{n-1}} = \frac{n!}{2^{n-1}} E[2^{X(s)}], \tag{2}$$

where $E[2^{X(s)}]$ denotes the expected value of the random variable $2^{X(s)}$ and s is a randomly chosen permutation of S_n.

2 The distribution of $X(s)$

Using the previous notation suppose, now, that $|\mathcal{T}| \geq \frac{n^2}{6} - O(n)$, and let us estimate the quantity (2) for large n. To do so, we show that X has an (approximately) Poisson distribution with mean 1 for large n.

Proposition 2 *For every $C > 0$, every integer $p \geq 0$ and every $\epsilon > 0$, there is an $n_0 = n_0(C, p, \epsilon)$ such that if $n \geq n_0$, $\mathcal{T}$ is a collection of $t \geq \frac{n^2}{6} - Cn$ pairwise edge-disjoint triangles in K_n, and if $X = X(s)$ is the random variable counting the number of members of $\mathcal{T}$ present in s, then*

$$|Pr[X = p] - \frac{1}{ep!}| < \epsilon.$$

Proof. We apply Brun's sieve in the form described, for example, in [2], Chapter 8. To do so, assign to each triangle $R \in \mathcal{T}$ an indicator random variable X_R, where $X_R = 1$ if and only if R is present in s. Let B_R be the event $X_R = 1$. Then $X = \sum_{R \in \mathcal{T}} X_R$ is the number of triangles present in s.

Denote

$$S^{(r)} = \sum Pr[B_{R_1} \wedge B_{R_2} \wedge \ldots B_{R_r}],$$

where the sum ranges over all subsets $\{R_1, R_2, \ldots, R_r\}$ of cardinality r of $\mathcal{T}$. By Brun's method it suffices to show that if $t \geq \frac{n^2}{6} - Cn$ and n is sufficiently large (as a function of C, r, ϵ) then

$$| \; S^{(r)} - \frac{1}{r!} \; | \leq \epsilon. \tag{3}$$

Fix a collection $\{R_1, R_2, \ldots, R_r\}$ of r distinct triangles in $\mathcal{T}$. If there is no vertex that lies in more than one member of the collection, then the number of permutations in which all R_i are present is precisely $6^r(n - 2r)!$.

It is not too difficult to check that in every other case, the number of permutations in which all R_i are present is smaller. Therefore, since clearly $t \leq n(n-1)/6 \leq n^2/6$,

$$S^{(r)} \leq \binom{t}{r} \frac{6^r(n-2r)!}{n!} \leq \frac{n^{2r}}{6^r r!} \frac{6^r}{n(n-1)(n-2)\ldots(n-2r+1)} \leq (1 + O(\frac{1}{n}))\frac{1}{r!}.$$

On the other hand, since $|\mathcal{T}| = t \geq \frac{n^2}{6} - Cn$, and since there are at most $(n-1)/2 < n/2$ members of $\mathcal{T}$ containing any given vertex, there are at least

$$\frac{t(t - 3n/2)(t - 6n/2)\ldots(t - 3(r-1)n/2)}{r!} \geq (1 - O(\frac{1}{n}))\frac{t^r}{r!}$$

collections consisting of r members of $\mathcal{T}$ with no vertex in more than one of them. This implies that

$$S^{(r)} \geq (1 - O(\frac{1}{n}))\frac{t^r}{r!}\frac{6^r(n-2r)!}{n!} = (1 - O(\frac{1}{n}))\frac{1}{r!}.$$

This completes the proof of (3) and hence implies the assertion of the proposition. ∎

Corollary 3 *For every $C > 0$ and $\delta > 0$ there is an $n_0 = n_0(\delta, C)$ such that for every $n > n_0$ and every collection $\mathcal{T}$ of $t \geq \frac{n^2}{6} - Cn$ pairwise edge-disjoint triangles in K_n, the random variable $X(s)$ defined in Section 1 satisfies*

$$E[2^{X(s)}] \geq e - \delta.$$

Proof. Since

$$\sum_{p=0}^{\infty} \frac{2^p}{ep!} = \frac{e^2}{e} = e,$$

there is some fixed $r = r(\delta)$ such that

$$\sum_{p=0}^{r} \frac{2^p}{ep!} \geq e - \frac{\delta}{2}.$$

By Proposition 2, if $n \geq n_0(\delta, C, r)$ then

$$Pr[X = p] > \frac{1}{ep!} - \frac{\delta}{2(r+1)2^p}$$

for all $p \leq r$. Therefore, for such an n,

$$E[2^{X(s)}] > \sum_{p=0}^{r} (\frac{1}{ep!} - \frac{\delta}{2(r+1)2^p})2^p = \sum_{p=0}^{r} \frac{2^p}{ep!} - \sum_{p=0}^{r} \frac{\delta}{2(r+1)} \geq e - \frac{\delta}{2} - \frac{\delta}{2} = e - \delta.$$

This completes the proof. ■

Proof of Theorem 1. The known results about the existence of Steiner Triple Systems, or the result in [4], imply that for every n there is a collection of $t \geq \frac{n^2}{6} - O(n)$ pairwise edge-disjoint triangles in K_n. Therefore, by Corollary 3 the quantity in (2) is at least $(e - o(1))\frac{n!}{2^{n-1}}$, where the $o(1)$-term tends to 0 as n tends to infinity. ■

3 Remarks

1. It is possible to obtain a lower bound for $P(n)$ for $n \geq 3$ by modifying the randomization scheme of Section 1 as follows. Let $\mathcal{T}$ be as before, and let $\mathcal{P}$ be a collection of p pairwise edge-disjoint pairs of connected edges in K_n where no edge lies in $\mathcal{P}$ and in $\mathcal{T}$. Orient each member of $\mathcal{T}$ in a random cyclic orientation, as before, and each member (i, j, k) (with j as the joint vertex) of $\mathcal{P}$, randomly and independently, either $(i, j), (j, k)$ or $(k, j), (j, i)$ with equal probability.

   For a permutation $s = (i_1, i_2, \ldots, i_n)$ of the vertices of T_n, and for a member (i, j, k) in $\mathcal{P}$, we say that it is present in s if either (i, j, k) or (k, j, i) appear in three consecutive places of s (in one of these two orders). Let $Y(s)$ denote the number of members in $\mathcal{P}$ which are present in s. Then the probability that s is a directed Hamiltonian path in the random tournament T_n is precisely $\frac{2^{X(s)+Y(s)}}{2^{n-1}}$. Therefore, the expected number of such paths in T_n is

$$\sum_{s \in S_n} \frac{2^{X(s)+Y(s)}}{2^{n-1}} = \frac{n!}{2^{n-1}} E[2^{X(s)+Y(s)}] \geq \frac{n!}{2^{n-1}} (1 + E[X(s) + Y(s)]), \tag{4}$$

where the preceeding inequality is due to the fact that $X(s), Y(s)$ are nonnegative integers and hence $2^{X(s)+Y(s)} \geq 1 + X(s) + Y(s)$ for every s.

Since each member of $\mathcal{T}$ is present in $6(n-2)!$ permutations and each member of $\mathcal{P}$ is present in $2(n-2)!$, we get that $E[2^{X(s)+Y(s)}] \geq \frac{n!}{2^{n-1}}(1 + \frac{6t+2p}{n(n-1)})$. Thus, considering (4) we have,

$$P(n) \geq \frac{n!}{2^{n-1}}(1 + \frac{6t+2p}{n(n-1)}). \tag{5}$$

It was established in [4] that it is possible to pack in K_n $\psi(n)$ pairwise edge-disjoint triangles where

$$\psi(n) = \begin{cases} \lfloor \frac{n}{3} \lfloor \frac{n-1}{2} \rfloor \rfloor & \text{for } n \neq 5 \pmod 6 \\ \lfloor \frac{n}{3} \lfloor \frac{n-1}{2} \rfloor \rfloor - 1 & \text{for } n = 5 \pmod 6 \end{cases}$$

It is also possible to pair some of the remaining edges to construct $\mathcal{P}$ with $p(n)$ pairs. By examining the proof in [4] it is not difficult to check that the best possible value of $p(n)$ given that we have already packed $\psi(n)$ edge disjoint triangles is $p(n) = 2$ for $n = 5$ mod 6, $p(n) = 1$ for $n = 4$ mod 6 and $p(n) = 0$ otherwise.

With these values of $\psi(n)$ and $p(n)$ our arguments thus give the following estimate for $P(n)$:

$$P(n) \geq \frac{n!}{2^{n-1}}(1 + \frac{6\psi(n) + 2p(n)}{n(n-1)}). \tag{6}$$

This can be used to derive some lower bounds for $P(n)$ which improve the estimate in (1) already for small values of n.

2. A *Hamiltonian cycle* in a tournament T is a spanning directed cycle in T. Applying Szele's [5] probability method to $C(n)$, the maximum number of directed Hamiltonian cycles in a tournament with n vertices, it is easy to show that $C(n) \geq \frac{(n-1)!}{2^n}$. It is also straightforward to apply our randomization scheme and analysis to improve the lower bound for $C(n)$ by a factor of about e. Simpler yet, is to apply a result in [1] which states that $C(n+1) \geq \frac{P(n)}{4}$ and derive this improvement from Theorem

1. The arguments above that yield (6), modified to deal with cycles, imply that for every $n > 2$

$$C(n) \geq \frac{(n-1)!}{2^n}(1 + \frac{6\psi(n) + 2p(n)}{(n-1)(n-2)}).$$

3. In his 1943 paper, Szele showed that $P(n) \leq O(\frac{n!}{2^{3n/4}})$ and conjectured that the limit, as n tends to infinity, of $(P(n)/n!)^{1/n}$ is $1/2$. This is proved in [1], where it is shown that $P(n) \leq O(n^{3/2}\frac{n!}{2^n})$ by applying some known results about permanents of $0,1$-matrices. Friedgut and Kahn [3] have recently found a clever refinement of the method of [1] and improved this upper bound by a factor of $n^{1/16}$. It would be interesting to decide if $P(n) = \Theta(\frac{n!}{2^n})$.

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