Choosability and fractional chromatic numbers

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Abstract

A graph G is (a, b)-choosable if for any assignment of a list of a colors to each of its vertices there is a subset of b colors of each list so that subsets corresponding to adjacent vertices are disjoint. It is shown that for every graph G, the minimum ratio a/b where a, b range over all pairs of integers for which G is (a, b)-choosable is equal to the fractional chromatic number of G.

1 Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E, and let $L(v)$ be a *list* of allowed colors assigned to each vertex $v \in V$. We say that G is *L-list-colorable* if there exists a coloring $c(v)$ of the vertices of G such that $c(v) \in L(v)$ for all $v \in V$ and $c(u) \neq c(v)$ for all edges $uv \in E$. Thus, list colorings are restricted types of proper vertex colorings. If G is L-list-colorable for every list assignment such that $|L(v)| = k$ for all $v \in V$, then G is called *k-choosable*. The *choice number*, $ch(G)$, is the smallest integer k for which G is k-choosable.

More generally, we say that G is (a, b)*-choosable* for some integers a and b, $a \geq 2b > 1$, if, for any assignment of lists with $|L(v)| = a$ for all $v \in V$, there are subsets $C(v) \subset L(v)$ with $|C(v)| = b$ such that $C(u)$ and $C(v)$ are disjoint for all pairs of adjacent vertices u and v.

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The concepts of list colorings and choosability were introduced in the 1970s by Vizing [9] and independently by Erdős, Rubin and Taylor [4]. Those early papers give an interesting introduction to the topic, including a lot of results and many open problems. Forgotten for more than a decade, some of the questions raised already in the seventies have been answered recently; see [2] for a survey and [7], [8] for more recent results and references. Still, a lot of intriguing open problems remain open, and in particular very little is known about the relationship between (a, b)-choosability and (c, d)-choosability where $(a, b) \neq (c, d)$. Erdős, Rubin and Taylor raised the following question.

Problem 1.1 *If G is (a, b)-choosable, does it follow that G is (am, bm)-choosable for every $m \geq 1$?*

Motivated by this problem, we consider the set

$$CH(G) := \{(a, b) : G \text{ is } (a, b)\text{-choosable}\}$$

and define the *choice ratio*

$$chr(G) := \inf\{a/b : (a, b) \in CH(G)\}.$$

Our aim is to prove that $chr(G)$ equals the so-called *fractional chromatic number* of G, a well-studied concept in polyhedral combinatorics, defined as follows. Denoting by $\mathcal{S}(G)$ the collection of all independent vertex sets in G, a *fractional coloring* is a mapping

$$\varphi : \mathcal{S}(G) \rightarrow \Re^{\geq 0}$$

such that

$$\sum_{\substack{S \in \mathcal{S}(G) \\ v \in S}} \varphi(S) \geq 1\,, \qquad \forall\ v \in V. \tag{1}$$

The *fractional chromatic number*, denoted $\chi^*(G)$, is the solution of the linear program (1) with objective function

$$\min \sum_{S \in \mathcal{S}(G)} \varphi(S). \tag{2}$$

Certainly, the minimum remains unchanged if the range of φ is restricted to the closed interval $[0, 1]$. Note further that the *chromatic number* $\chi(G)$ is obtained when we view (1), (2) as a *discrete* optimization problem, i.e. with $\varphi : \mathcal{S}(G) \rightarrow \{0, 1\}$.

While the choice number of a graph can be much larger than its chromatic number (e.g. $ch(K_{n,n}) = \Theta(\log n)$, cf. [4]), Gutner ([6], cf. also [2], Proposition 4.6) has proven (using a different terminology) that the choice ratio $chr(G)$ never exceeds the chromatic number $\chi(G)$ of G. Here we prove the following stronger result.

Theorem 1.2 *The choice ratio of any graph $G = (V, E)$ equals its fractional chromatic number.*

This result may be viewed as the pair of the two inequalities $chr(G) \geq \chi^*(G)$ and $\chi^*(G) \geq chr(G)$. In fact, the latter can also be strengthened, by showing that the infimum can be replaced by minimum in the definition of $chr(G)$.

Theorem 1.3 *For every integer n there exists a number $f(n) \leq (n+1)^{2n+2}$ such that the following holds. For every graph G with n vertices and with fractional chromatic number χ^*, and for every integer M which is divisible by all integers up to $f(n)$, G is $(M, M/\chi^*)$-choosable.*

We also note that for every G and M as above, M/χ^* is an integer, as shown by the observations in Section 2.

The paper is organized as follows. In the next section some known properties of the fractional chromatic number are recalled, and in Section 3 two lemmas on partitions of sequences and uniform hypergraphs are given. Theorems 1.2 and 1.3 are proved in Section 4. The proof of the former is probabilistic, while that of the latter combines some of the techniques in [3] and in [2] with some additional ideas. Finally, in Section 5 we show that every cycle of length $2t + 1$ is $(2t + 1, t)$-choosable. This example indicates that the smallest M for which G is $(M, M/\chi^*)$-choosable may be much smaller than the bound given in Theorem 1.3.

2 Some properties of χ^*

The linear inequalities (1) together with the conditions

$$\varphi(S) \geq 0\,, \qquad \forall\ S \in \mathcal{S}(G) \tag{3}$$

describe a convex body $\mathcal{P}$ in $\Re^{|\mathcal{S}(G)|}$ on which the objective function (2) attains its minimum at some (at least one) vertex. This vertex can be described as the intersection of $|\mathcal{S}(G)|$ facets of $\mathcal{P}$, all but at most n of which are of the form $\varphi(S) = 0$. Thus, there exists a subfamily $\mathcal{S}_0 \subseteq \mathcal{S}(G)$, $|\mathcal{S}_0| \leq n$, and positive reals $\{w_S : S \in \mathcal{S}_0\}$ such that

$$\chi^*(G) = \sum_{S \in \mathcal{S}_0} w_S$$

where the w_S are the solutions of the corresponding system of linear equations. This implies, by Cramer's rule, that all the w_S are of the form p_S/q, where the $p_S > 0$ are integers, and q is the absolute value of a determinant of an n by n matrix A with $0, 1$ entries. The following bound on the absolute value of the determinant of such a matrix is well known.

Lemma 2.1 *With the above notation,* $|detA| \leq 2^{-n}(n+1)^{(n+1)/2}$.

Proof. Embed the matrix A in the lower right corner of an $(n+1)$ by $(n+1)$ matrix B whose first row is the all-1 vector and whose first column is the (column) vector $(1, 0, 0, \ldots, 0)$. Note that the determinant $detB$ of B equals that of A. Now, multiply all the rows of B but the first one by 2, and subtract the first row from all the others to get an $(n+1)$ by $(n+1)$ matrix C in which all entries are $+1$ or -1. Then $detC = 2^n detB = 2^n detA$ and, by Hadamard's Inequality, $|detC| \leq (n+1)^{(n+1)/2}$ as needed. $\square$

Defining

$$p := \sum_{S \in \mathcal{S}_0} p_S,$$

we have $\chi^*(G) = p/q$ with

$$p \leq nq \leq n2^{-n}(n+1)^{(n+1)/2} \leq (n+1)^{(n+1)/2}.$$

Moreover, taking each $S \in \mathcal{S}_0$ with multiplicity p_S, we obtain a collection $\{S_1, S_2, \ldots, S_p\}$ of not necessarily distinct independent sets in G so that every vertex lies in precisely q of those S_i. Such a (multi)set of independent subsets provides an equivalent interpretation of an optimal fractional coloring.

3 Uniform partitions

In the proof of Theorem 1.3 we shall need two lemmas which we present in this section. The first one is the following simple observation.

Lemma 3.1 *Let* $(n_i : i \in I)$ *be a sequence of positive integers, where each* n_i *is at most* k*. Let* M *and* p *be two integers and suppose that* $\sum_{i \in I} n_i = M$*, that* M/p *is divisible by all integers up to* k*, and that* $k \cdot lcm(2, 3, \ldots, k) \leq M/p$*, where* $lcm(2, 3, \ldots, k)$ *denotes the least common multiple of* $2, 3, \ldots, k$*. Then, there is a partition* $I = I_1 \cup I_2 \cup \cdots \cup I_p$ *of* I *into* p *pairwise disjoint sets such that for every* j*,* $1 \leq j \leq p$*,* $\sum_{i \in I_j} n_i = M/p$.

Proof. Put $m = lcm(2, 3, \ldots, k)$. First, we shall construct a large number of pairwise disjoint subsets J_s of I so that $\sum_{i \in J_s} n_i = m$ for every s, from which the classes of the required partition will be created. As long as there are at least m/t numbers n_i equal to t for some t, $1 \leq t \leq k$, among the elements n_i which have not been used yet, form a new subset consisting of the indices of m/t such elements and remove them from the sequence. When this process terminates, we are left with at most $m/t - 1$ occurrences of t among the remaining elements, and hence the sum of all remaining elements is smaller than $\sum_{t=1}^{k}(m/t)t = km \leq M/p$. All the other elements of the sequence have been partitioned into subsets, in such a way that the sum of elements in each subset is precisely m. By assumption, m divides M/p, hence $r = \frac{M}{mp}$ is an integer. Note that we have at least $(p-1)r$ subsets J_s as above, since the total sum of elements in these subsets is at least $M - M/p = (p-1)rm$. We can thus form from these subsets $p-1$ collections consisting of r subsets each, and take all the elements in the remaining subsets together with the elements in no subsets as the final collection, completing the proof. $\square$

Remark. The assumption that $M/p \geq k \cdot lcm(2, 3, \ldots, k)$ can, in fact, be dropped, using the argument in [3]. Since, however, this is not crucial here and makes the proof slightly more complicated, we omit the details.

A hypergraph $\mathcal{H} = (X, \mathcal{F})$ is called *ℓ-uniform* if each of its edges contains precisely ℓ vertices. If R is a subset of the vertex set of $\mathcal{H}$, let $\mathcal{H}_R$ denote the hypergraph with vertex set R and edge set $\{F \cap R : F \in \mathcal{F}\}$. The hypergraph is (uniformly) *reducible* if there is a two-coloring of its vertex set, $X = R \cup B$, $R \cap B = \emptyset$, so that the hypergraph $\mathcal{H}_R$ is ℓ'-uniform for some $0 < \ell' < \ell$. Note that, in this case, the hypergraph $\mathcal{H}_B$ is $(\ell - \ell')$-uniform. Huckemann, Jurkat and Shapley (cf. [5]; see also [1] for another proof with a slightly worse estimate) proved that if $\ell \geq (n+1)^{(n+1)/2}$, then every ℓ-uniform hypergraph with n edges is reducible. Applying straightforward induction, this implies the following.

Lemma 3.2 *Let $\mathcal{H} = (X, \mathcal{F})$ be a uniform hypergraph with n edges. Then there is a partition $X = \bigcup_{i \in I} X_i$ of X into pairwise disjoint sets such that $\mathcal{H}_{X_i}$ is n_i-uniform and $n_i \leq (n+1)^{(n+1)/2}$ for every $i \in I$.* $\square$

4 Proof of the main results

In this section we prove Theorems 1.2 and 1.3, the former in two parts. We begin with the lower bound on $chr(G)$.

Proof of the lower bound $chr(G) \geq \chi^*(G)$.

Let $(a, b) \in CH(G)$ be arbitrary. We have to show that $\chi^*(G) \leq a/b$. For this purpose we define identical lists

$$L(v) := \{1, 2, \ldots, a\}$$

for all $v \in V$. By assumption, some b-element subsets $C(v) \subset L(v)$ can be chosen such that $C(u) \cap C(v) = \emptyset$ for all $uv \in E$. Set

$$S_i := \{v \in V : i \in C(v)\}, \qquad i = 1, 2, \ldots, a$$

and, for every $S \in \mathcal{S}(G)$, define

$$\varphi(S) := a_i/b$$

where a_i is the number of occurrences of S in the multiset $\{S_1, \ldots, S_a\}$. By the conditions on the subsets $C(v)$, each of the a sets S_i is independent, and each vertex appears in precisely b of them. Thus, φ is a fractional coloring with value a/b. $\square$

We now turn to the proof of the converse inequality and to that of Theorem 1.3. Throughout, we assume that a collection of a-element lists $L(v)$ is given for the vertices v of the graph $G = (V, E)$, and denote

$$L := \bigcup_{v \in V} L(v).$$

Moreover, based on the observations given in the preceding sections, we assume in either case that $\chi^*(G) = p/q$, $q \leq (n+1)^{(n+1)/2}$, and that $S_1, S_2, \ldots, S_p \in \mathcal{S}(G)$ are p (not necessarily distinct) independent sets of G such that each vertex is contained in precisely q of the S_i. Though the assertion of Theorem 1.3 implies the required converse inequality, nevertheless we include a separate proof of this converse since it is simpler and applies to somewhat smaller values of a and b.

Proof of the upper bound $chr(G) \leq \chi^*(G)$**.**

We have to show that for every $\varepsilon > 0$ there exists a pair $(a, b) \in CH(G)$ such that $a \leq (1+\varepsilon)b \cdot \chi^*(G)$. Consider $a := (1+\varepsilon)pm$ and $b := qm$, for m sufficiently large, where we assume for simplicity (and without loss of generailty) that both a and b are integers. Take a random partition

$$L = L_1 \cup \cdots \cup L_p$$

where $Prob(i \in L_j) = 1/p$ for every $1 \leq j \leq p$, independently for all $i \in L$. Noting that p and $n := |V|$ are fixed for any given G, it follows from well-known estimates on the binomial distribution that, for each $v \in V$ and each $j \in \{1, \ldots, p\}$,

$$|L(v) \cap L_j| = m + \varepsilon m - o(m) \tag{4}$$

holds with probability greater than $1 - \frac{1}{np}$ for all sufficiently large m. Thus, with positive probability, (4) holds simultaneously for all v and all j. Assuming that $L_1 \cup \cdots \cup L_p$ is a suitable partition of L satisfying $|L(v) \cap L_j| \geq m$ for all v and j, we can choose m-element subsets $C_j(v) \subseteq L(v) \cap L_j$. Then, clearly, the b-element sets

$$C(v) := \bigcup_{1 \leq j \leq p} C_j(v)$$

satisfy the requirements for (a, b)-choosability, implying $(a, b) \in CH(G)$. $\square$

Proof of Theorem 1.3. Suppose that M is divisible by all integers up to, say, $(n+1)^{2n+2}$, and consider the collection $(L(v) : v \in V)$ of lists with $a = M$ colors each, assigned to the vertices of G. Let $\mathcal{H} = (X, \mathcal{F})$ be the hypergraph whose vertex set is L, the set of all colors in all the lists, and whose edge set is the set of the n lists $L(v)$. Define $k = (n+1)^{(n+1)/2}$. By Lemma 3.2, the vertex set X can be partitioned into pairwise disjoint sets $(X_i : i \in I)$, such that $\mathcal{H}_{X_i}$ is n_i-uniform, with $n_i \le k$ for every i.

The initial assumptions and the fact that $p \le (n+1)^{(n+1)/2}$ imply that M/p is divisible by all numbers up to k (since M is divisible by pr for all $r \le k$) and that M/p is bigger than $k \cdot lcm(2, 3, \ldots, k)$ (since M is divisible by pkr for all $r \le k$). Therefore, by Lemma 3.1, there is a partition $I = I_1 \cup I_2 \cup \cdots \cup I_p$ such that $\sum_{i \in I_j} n_i = M/p$ for every j, $1 \le j \le p$. Define, now, for each vertex v of G a subset $C(v)$ of $L(v)$ by

$$C(v) := \bigcup \, \{L(v) \cap X_i : i \in I_j \text{ for some } j \text{ for which } v \in T_j\}.$$

There are precisely q values of j for which v is in T_j, and each such j contributes exactly $\sum_{i \in I_j} n_i = M/p$ colors to $C(v)$, giving a total of $Mq/p = M/\chi^*$ colors for each vertex. Since each color lies in the sets $C(v)$ for all members v of some independent set in the collection $\{S_1, \ldots, S_p\}$, it follows that the sets $C(u)$ and $C(v)$ are disjoint for each pair of adjacent vertices u and v. This completes the proof of the theorem. $\square$

5 An example: odd cycles

We end this note by showing that for odd cycles C_{2r+1} the smallest value of M for which the cycle is $(M, M/\chi^*)$-choosable is not very large. It is immediately seen that $\chi^*(C_{2r+1}) = 2 + 1/r$.

Proposition 5.1 *Every odd cycle C_{2r+1} is $(2r+1, r)$-choosable.*

Proof. Let $v_1, \ldots, v_{2r+1}$ be the vertices of the cycle C_{2r+1}, and let $(L(v_i) : 1 \le i \le 2r+1)$ be the collection of the $(2r+1)$-element lists of allowed colors assigned to the vertices. Denote by $S := \{f_1, \ldots, f_t\} := \bigcap_{v_i \in V} L(v_i)$ the intersection of all these lists. Clearly, $t \le 2r+1$ by the assumption $|L(v_i)| = 2r+1$. First, we generate new lists $L'(v_i) := L(v_i) \setminus \{f_i\}$ for $i = 1, \ldots, t$ and $L'(v_i) := L(v_i)$ for $t < i \le 2r+1$. Note that the new lists have at least $2r$ elements each, and that no color belongs to all of them. Next, we orient the edges of C_{2r+1} clockwise, to obtain a directed cycle.

Now, we can choose r colors for every vertex in the following way. Take an arbitrary color f appearing in one of the lists and consider the subgraph G_f of C_{2r+1} induced by all vertices which contain f in their lists. Every such subgraph is the union of directed paths, therefore contains an independent set S_f such that every $v \in V(G_f)$ either is in

S_f or has its successor in S_f. We choose the color f for all vertices of S_f, and delete f from all lists. If we have already chosen r colors for a vertex, we remove this vertex from the graph. In this way we delete a color f from the list of a vertex v only if we choose f either for v itself or for its successor. Thus, we can choose r colors for every vertex. $\square$

By the same argument, the following more general assertion can also be proved.

Proposition 5.2 *The cycle C_{2r+1} is $(2t+1, t)$-choosable for every t, $1 \leq t \leq r$.*

The condition $t \leq r$ above is necessary, by the inequality $chr(G) \geq \chi^*(G)$.

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