On Partitions of Discrete Boxes

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Abstract

We prove that any partition of an n-dimensional discrete box into nontrivial sub-boxes must consist of at least 2^n sub-boxes, and consider some extensions of this theorem.

1 The theorem

A set of the form

$$A = A_1 \times A_2 \times \cdots \times A_n,$$

where $A_1, A_2, \ldots, A_n$ are finite sets with $|A_i| \geq 2$, will be called here an *n-dimensional discrete box.* A set of the form $B = B_1 \times B_2 \times \cdots \times B_n$, where $B_i \subseteq A_i$, $i = 1, \ldots, n$, is a *sub-box* of A. Such a set B is said to be *nontrivial* if $\emptyset \neq B_i \neq A_i$ for every i.

The following theorem answers a question posed by Kearnes and Kiss [1, Problem 5.5].

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Theorem 1 *Let A be an n-dimensional discrete box, and let $\{B^1, B^2, \ldots, B^m\}$ be a partition of A into nontrivial sub-boxes. Then $m \geq 2^n$.*

Proof. Let

$$B^j = B_1^j \times B_2^j \times \cdots \times B_n^j, \ j = 1, \ldots, m.$$

Let us call a sub-box C of A odd if its cardinality is odd. Let $\mathcal{O}(A)$ denote the collection of all odd sub-boxes of A. For $j = 1, \ldots, m$, define:

$$\mathcal{O}_j(A) = \{C \in \mathcal{O}(A) \mid C \cap B^j \text{ is odd}\}.$$

A sub-box is odd if and only if each of its n factors has odd cardinality, and the nontriviality of the B^j implies that half of the odd cardinality subsets of A_i intersect B_i^j in an odd number of elements. This implies

$$\frac{|\mathcal{O}_j(A)|}{|\mathcal{O}(A)|} = \frac{1}{2^n}, \ j = 1, \ldots, m. \tag{1}$$

For each $C \in \mathcal{O}(A)$ the partition $\{B^1, B^2, \ldots, B^m\}$ induces a partition of C in which at least one of the parts must have odd cardinality, which implies

$$\bigcup_{j=1}^{m} \mathcal{O}_j(A) = \mathcal{O}(A). \tag{2}$$

It follows from (1) and (2) that $m \geq 2^n$.

2 Extensions and non-extensions

2.1 Infinite boxes

The theorem remains true if in the definition of an n-dimensional discrete box we allow the sets $A_1, A_2, \ldots, A_n$ to be infinite. This follows by considering the finitely many atoms induced by the partition at hand.

2.2 Partitions mod 2

The theorem remains true, with the same proof, if $\{B^1, B^2, \ldots, B^m\}$ is only assumed to be a partition mod 2, that is, $\{B^1, B^2, \ldots, B^m\}$ is a multi-family of nontrivial sub-boxes of A such that every point of A is covered an odd number of times.

2.3 Conditions for equality

An obvious example of equality in the theorem is obtained by splitting each A_i into two nonempty parts, and taking $B^1, B^2, \ldots, B^{2^n}$ to be the corresponding cells. One can derive from the above proof some conditions that any example of equality must satisfy, and one might hope that these will lead to a characterization of all such examples. In particular, one might naively conjecture that every n-dimensional example of equality may be obtained by splitting one factor into two parts, and further partitioning each of the two resulting boxes according to some $(n-1)$-dimensional examples of equality. However, the following partition of a $3 \times 3 \times 3$ box into 8 nontrivial sub-boxes, in which none of the factors is split into just two parts, seems to indicate that examples of equality do not obey a simple construction rule:

$$\begin{array}{lcccccc}
A & = & \{1,2,3\} & \times & \{a,b,c\} & \times & \{\alpha,\beta,\gamma\} \\
B^1 & = & \{1\} & \times & \{a\} & \times & \{\alpha\} \\
B^2 & = & \{1\} & \times & \{a\} & \times & \{\beta,\gamma\} \\
B^3 & = & \{1\} & \times & \{b,c\} & \times & \{\alpha,\beta\} \\
B^4 & = & \{1,2\} & \times & \{b,c\} & \times & \{\gamma\} \\
B^5 & = & \{2,3\} & \times & \{a,b\} & \times & \{\alpha,\beta\} \\
B^6 & = & \{2,3\} & \times & \{a\} & \times & \{\gamma\} \\
B^7 & = & \{2,3\} & \times & \{c\} & \times & \{\alpha,\beta\} \\
B^8 & = & \{3\} & \times & \{b,c\} & \times & \{\gamma\}
\end{array}$$

2.4 Partition numbers of hypergraphs

If $\mathcal{H} = (V, E)$ is a hypergraph (i.e., E is a family of subsets of V), let us define the *partition number* $\pi(\mathcal{H})$ as the least p such that E contains a partition $\{B^1, B^2, \ldots, B^p\}$ of V (letting $\pi(\mathcal{H}) = \infty$ if there is no such p). If $\mathcal{H}_1 = (V_1, E_1)$ and $\mathcal{H}_2 = (V_2, E_2)$ are two hypergraphs, let us define their *product* $\mathcal{H}_1 \times \mathcal{H}_2$ to be the hypergraph with vertex-set $V_1 \times V_2$ and edge-set consisting of all sets of the form $B_1 \times B_2$, $B_1 \in E_1$, $B_2 \in E_2$.

Clearly, if E consists of all the proper subsets of V and $|V| \geq 2$, then the partition number of $\mathcal{H} = (V, E)$ is 2. Our theorem asserts that the product of n such hypergraphs has partition number 2^n. This raises the question whether the partition number is multiplicative with respect to hypergraph product. It is easy to see that $\pi(\mathcal{H}_1 \times \mathcal{H}_2) \leq \pi(\mathcal{H}_1) \cdot \pi(\mathcal{H}_2)$, but the following example shows that in general equality need not hold.

Let $k > 4$ be an integer, and let V_1 and V_2 be two sets of cardinality $3k$. Let E_1 consist of all subsets of V_1 of cardinality 1 or $k+1$, and let E_2 consist of all subsets of V_2 of cardinality 1 or $2k-1$. Then $\mathcal{H}_1 = (V_1, E_1)$ and $\mathcal{H}_2 =$

(V_2, E_2) satisfy $\pi(\mathcal{H}_1) = k$ and $\pi(\mathcal{H}_2) = k + 2$. However, $\pi(\mathcal{H}_1 \times \mathcal{H}_2) \leq 6k$. In order to see this, identify the vertex-set of $\mathcal{H}_1 \times \mathcal{H}_2$ with the edge-set $E(K_{3k,3k})$ of a complete bipartite graph with $3k$ vertices on each side. Find a $(k+1)$-regular subgraph G of $K_{3k,3k}$, and partition the edge-sets of G and its bipartite complement into $3k$ stars each, centered on opposite sides. As $6k < k(k + 2)$ for $k > 4$, this is a counterexample to the multiplicativity of the partition number with respect to hypergraph product.

One may define the *mod 2 partition number* $\overline{\pi}(\mathcal{H})$ in a similar way, by considering partitions mod 2 (as in subsection 2.2) instead of partitions. Here, too, multiplicativity fails in general. Let $\mathcal{H}_1 = (V_1, E_1)$ and $\mathcal{H}_2 = (V_2, E_2)$ be two copies of a Fano plane (vertices are points, edges are lines). Then $\overline{\pi}(\mathcal{H}_i) = 3$ for $i = 1, 2$, but $\overline{\pi}(\mathcal{H}_1 \times \mathcal{H}_2) \leq 7$, as shown by the mod 2 partition of $V_1 \times V_2$ formed by taking the product of each line with itself.

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References

[1] K. A. Kearnes and E. W. Kiss, Finite algebras of finite complexity, *Discrete Math.* **207** (1999), 89-135.