

# Packing and Covering Dense Graphs

Noga Alon \*

Yair Caro †

Raphael Yuster ‡

## Abstract

Let  $d$  be a positive integer. A graph  $G$  is called  $d$ -divisible if  $d$  divides the degree of each vertex of  $G$ .  $G$  is called nowhere  $d$ -divisible if no degree of a vertex of  $G$  is divisible by  $d$ . For a graph  $H$ ,  $\gcd(H)$  denotes the greatest common divisor of the degrees of the vertices of  $H$ . The  $H$ -packing number of  $G$  is the maximum number of pairwise edge disjoint copies of  $H$  in  $G$ . The  $H$ -covering number of  $G$  is the minimum number of copies of  $H$  in  $G$  whose union covers all edges of  $G$ . Our main result is the following:

For every fixed graph  $H$  with  $\gcd(H) = d$ , there exist positive constants  $\epsilon(H)$  and  $N(H)$  such that if  $G$  is a graph with at least  $N(H)$  vertices and has minimum degree at least  $(1 - \epsilon(H))|G|$ , then the  $H$ -packing number of  $G$  and the  $H$ -covering number of  $G$  can be computed in polynomial time. Furthermore, if  $G$  is either  $d$ -divisible or nowhere  $d$ -divisible, then there is a *closed formula* for the  $H$ -packing number of  $G$ , and the  $H$ -covering number of  $G$ .

Further extensions and solutions to related problems are also given.

## 1 Introduction

All graphs considered here are finite, undirected and simple, unless otherwise noted. For the standard graph-theoretic terminology the reader is referred to [1]. Let  $H$  be a graph without isolated vertices. An  $H$ -covering of a graph  $G$  is a set  $L = \{G_1, \dots, G_s\}$  of subgraphs of  $G$ , where each subgraph is isomorphic to  $H$ , and every edge of  $G$  appears in at least one member of  $L$ . The  $H$ -covering number of  $G$ , denoted by  $C(H, G)$ , is the minimum cardinality of an  $H$ -covering of  $G$ . An  $H$ -packing of a graph  $G$  is a set  $L = \{G_1, \dots, G_s\}$  of edge-disjoint subgraphs of  $G$ , where each subgraph is isomorphic to  $H$ . The  $H$ -packing number of  $G$ , denoted by  $P(H, G)$ , is the maximum cardinality of an  $H$ -packing of  $G$ .  $G$  has an  $H$ -decomposition if it has an  $H$ -packing which is

---

\*Department of Mathematics, Tel-Aviv University, Israel, e-mail: noga@math.tau.ac.il. Research supported in part by a USA Israeli BSF grant and by a grant from the Israel Science Foundation.

†Department of Mathematics, University of Haifa-ORANIM, Tivon 36006, Israel. e-mail: zeac603@uvm.haifa.ac.il

‡Department of Mathematics, University of Haifa-ORANIM, Tivon 36006, Israel. e-mail: raphy@math.tau.ac.il

also an  $H$ -covering. Recently, exact formulas for  $P(H, K_n)$  and  $C(H, K_n)$  have been obtained, for  $n \geq n(H)$  [3, 4]. A main tool which is used in both these papers is the result of Gustavsson [10] concerning the decomposition of dense graphs. In case the graph  $G$  is not complete, it is known that computing  $P(H, G)$  and  $C(H, G)$  is, in general, NP-Hard, as shown by Dor and Tarsi [7]. The purpose of this paper is to extend Gustavsson's result, and the above mentioned results for  $K_n$  and show that if the graph  $G$  is very dense, then the  $H$ -packing and  $H$ -covering numbers of  $G$  can be determined in polynomial time, and in many cases, these numbers can be given by a closed formula. To describe our results we need the following definitions. A graph  $G$  is called  $d$ -divisible if the degree of each vertex of  $G$  is a multiple of  $d$ .  $G$  is called *nowhere  $d$ -divisible* if no vertex of  $G$  has degree which is a multiple of  $d$ . Note that regular graphs are either  $d$ -divisible or nowhere  $d$ -divisible for every  $d$ . Also note that every graph is 1-divisible. For a graph  $H$  let  $\gcd(H)$  denote the greatest common divisor of the degrees of the vertices of  $H$ . For example,  $\gcd(C_4) = 2$ , whereas  $\gcd(T) = 1$  for every tree  $T$ . Our main results are the following:

**Theorem 1.1 (Packing Dense Graphs)** *Let  $H$  be a graph with  $h$  edges, and let  $\gcd(H) = d$ . Then there exist  $N = N(H)$ , and  $\epsilon = \epsilon(H)$  such that if  $G = (V, E)$  is a graph with  $n > N(H)$  vertices and  $\delta(G) > (1 - \epsilon(H))n$ , then  $P(H, G)$  can be determined in polynomial time. Furthermore, if  $G$  is either  $d$ -divisible or nowhere  $d$ -divisible then,*

$$P(H, G) = \lfloor \frac{\sum_{v \in V} \alpha_v}{2h} \rfloor,$$

where  $\alpha_v$  is the degree of vertex  $v$ , **rounded down** to the closest multiple of  $d$ . The r.h.s. of this formula should be reduced by 1 if  $G$  is  $d$ -divisible and  $0 < |E| \bmod h \leq d^2/2$ .

**Theorem 1.2 (Covering Dense Graphs)** *Let  $H$  be a graph with  $h$  edges, and let  $\gcd(H) = d$ . Then there exist  $N = N(H)$ , and  $\epsilon = \epsilon(H)$  such that if  $G = (V, E)$  is a graph with  $n > N(H)$  vertices and  $\delta(G) > (1 - \epsilon(H))n$ , then  $C(H, G)$  can be determined in  $O(n^{2.5})$  time. Furthermore, if  $G$  is either  $d$ -divisible or nowhere  $d$ -divisible then,*

$$C(H, G) = \lceil \frac{\sum_{v \in V} \alpha_v}{2h} \rceil,$$

where  $\alpha_v$  is the degree of vertex  $v$ , **rounded up** to the closest multiple of  $d$ . The r.h.s. of this formula should be increased by 1 if  $G$  is  $d$ -divisible and  $h$  divides  $|E| + d/2$ .

This paper is organized as follows. In section 2 we describe the tools and prove some lemmas which will be used in the proofs of Theorems 1.1 and 1.2. In section 3 we prove Theorem 1.1. In section 4 we prove Theorem 1.2. The final section contains results and extensions which solve some

variations of the packing and covering problems, among them are the *leave* and *excess* problems (see, e.g. [6] pages 263-264 and pages 411-412, and [11, 12]), and the efficient 2-overlap covering problem of Etzion [5, 2].

Throughout this sequel we use the notation  $e(G)$  to denote the number of edges of a graph  $G$ .  $d_G(v)$  denotes the degree of vertex  $v$  in  $G$ . For  $X \subset V$ ,  $G[X]$  denotes the subgraph of  $G = (V, E)$  induced by  $X$ .

## 2 Preparing the tools

As mentioned in the introduction, our main tool is the following result of Gustavsson [10]:

**Lemma 2.1 (Gustavsson's Theorem [10])** *Let  $H$  be a graph with  $h$  edges. There exists  $N_0 = N_0(H)$ , and  $\gamma = \gamma(H) > 0$ , such that for all  $n > N_0$ , if  $G$  is a graph on  $n$  vertices and  $m$  edges, with  $\delta(G) \geq n(1 - \gamma)$ ,  $\gcd(H) \mid \gcd(G)$ , and  $h \mid m$ , then  $G$  has an  $H$ -decomposition.  $\square$*

It is worth mentioning that  $N_0(H)$  in Gustavsson's Theorem is a rather huge constant; in fact, it is a highly exponential function of  $h$ . Also, the  $\gamma(H)$  is very small; in fact it is less than  $10^{-24}h^{-1}$ . Thus, the graph  $G$  needs to be large and dense, but it may still be far from being complete (i.e.,  $\Omega(n^2)$  edges may be missing).

The following lemma is a cornerstone in the proofs of Theorems 1.1 and 1.2. In essence, it shows that dense graphs contain sparse spanning subgraphs with predetermined degrees. In fact, this lemma can be viewed as an approximate bounded-degree version of the  $f$ -Factor Theorem of Tutte [13].

**Lemma 2.2** *Let  $d$  be a positive integer, and let  $G = (V, E)$  be a graph on  $n$  vertices with  $\delta(G) \geq n - n/(4d + 2) + 2d + 1$ . Let  $\{\nu_v \mid v \in V\}$  be a set of positive integers not exceeding  $2d$  whose sum is even. Then  $G$  contains a spanning subgraph  $G^*$  in which the degree of each vertex  $v$  is  $\nu_v$ .*

**Proof:** For  $i = 1, \dots, 2d + 1$ , let  $A^i$  be a (0-1)-sequence indexed by  $V$ , where  $A_v^i = 1$  if  $i \leq \nu_v$ , and  $A_v^i = 0$  if  $i > \nu_v$ . Clearly,  $A^1$  has all its elements equal to 1, while  $A^{2d+1}$  has all its elements equal to 0. We call a sequence  $A^i$  *odd* if it contains an odd number of ones. Consider all the odd sequences  $A^i$  for  $2 \leq i \leq 2d$ , having at most  $n/2$  elements equal to 1. We may pick from each such sequence one location which is zero, such that no two sequences picked the same location. This can be done since the number of sequences is at most  $2d$ , while the number of zeroes in each is at least  $n/2$ , and  $n/2 > 2d$ . We modify the location picked for  $A^i$  to 1, and the corresponding location of  $A^1$  is set to 0. Now consider all the odd sequences  $A^i$  for  $2 \leq i \leq 2d$ , having more than  $n/2$  elements equal to 1. We may pick from each such sequence one location which equals 1,

such that no two sequences picked the same location. We modify the location picked for  $A^i$  to 0, and the corresponding location of  $A^{2d+1}$  is set to 1. This process guarantees that all the sequences  $A^2, \dots, A^{2d}$  have an even number of ones. If  $A^1$  also has an even number of ones after the process is completed, then so does  $A^{2d+1}$ . Otherwise, both  $A^1$  and  $A^{2d+1}$  have an odd number of ones, and the number of ones in  $A^1$  is at least  $n - 2d$  while the number of ones in  $A^{2d+1}$  is at most  $2d$ . Since  $n > 4d$ , we have a location which is 1 in  $A^1$  and 0 in  $A^{2d+1}$ , so we may switch the value of this location in both  $A^1$  and  $A^{2d+1}$ , and we have that all sequences have an even number of ones.

We now wish to make the number of ones in any pair of sequences differ by at most 2, while maintaining an even number of ones in each sequence. The following shifting procedure achieves this. If the number of ones in  $A^i$  and  $A^j$  differ by more than two (assume  $A^i$  has more ones than  $A^j$ ), we have two locations that contain 1 in  $A^i$  and 0 in  $A^j$ . by switching the values in these locations in both  $A^i$  and  $A^j$ , the number of ones in  $A^j$  is now closer to the number of ones in  $A^i$ , and they still both have an even number of ones. We continue with this procedure until the number of ones in any pair of sequences differ by at most 2.

The total number of ones in all the sequences is at least  $n$ , and therefore each sequence contains at least  $n/(2d+1) - 2$  ones. Also note that for each  $v \in V$ ,

$$\sum_{i=1}^{2d+1} A_v^i = \nu_v.$$

We associate with each sequence  $A^i$ , a matching  $M^i$  of  $G$  in the following way. The set of vertices matched in  $M^i$  is exactly the set of vertices which correspond to locations having the value 1 in  $A^i$ . Furthermore, each pair of matchings is edge disjoint. We need to show that, indeed, we can produce the set of matchings  $M^1, \dots, M^{2d+1}$ . Assume that we have already produced  $M^1, \dots, M^i$ . We show how to produce  $M^{i+1}$ . Let  $G'$  be the subgraph of  $G$  obtained by deleting the edges of  $M^1 \cup \dots \cup M^i$ , and then deleting the vertices whose corresponding location in  $M^{i+1}$  is 0. We need to show that  $G'$  has a perfect matching, since we can take such a matching as  $M^{i+1}$ . Let  $r$  denote the number of ones in  $A^{i+1}$  (note that  $r \geq n/(2d+1) - 2$  is even and is also the number of vertices of  $G'$ ). It suffices to show that  $\delta(G') \geq r/2$ , since this guarantees the existence of a perfect matching. Indeed,

$$\begin{aligned} \delta(G') &\geq r - 1 - (n - 1 - \delta(G)) - i \geq r - n + \delta(G) - 2d \geq \\ &r - n + (n - n/(4d+2) + 2d + 1) - 2d \geq r - n/(4d+2) + 1 \geq r/2. \end{aligned}$$

The lemma now follows from the fact that the union of all the matchings is a spanning subgraph  $G^*$  of  $G$  with the property that each vertex  $v$  has degree  $\nu_v$  in  $G^*$ .  $\square$

### 3 Packing dense graphs

In this section we prove Theorem 1.1. Given  $H$ , we choose

$$N(H) = \max\{N_0(H), \frac{4d}{\gamma(H)}, 1000h^5\}$$

where  $N_0(H)$  and  $\gamma(H)$  are as in Lemma 2.1. We also choose

$$\epsilon(H) = \min\{\frac{\gamma(H)}{2}, \frac{1}{100d^4}, \frac{1}{2h}\}.$$

Now let  $G$  be a graph with  $n > N(H)$  vertices and  $\delta(G) > (1 - \epsilon(H))n$ . We need to show how  $P(H, G)$  can be computed in polynomial time, and, moreover, supply a closed formula for  $P(H, G)$  in case  $G$  is either  $d$ -divisible or nowhere  $d$ -divisible.

Let  $0 \leq b < 2h/d$  satisfy  $(\sum_{v \in V} \alpha_v)/d \equiv b \pmod{2h/d}$ . Note that since  $d = \gcd(H)$  and since  $2h$  is the sum of the degrees of the vertices of  $H$ , then  $2h/d$  must be an integer. Also note that  $\sum_{v \in V} (\alpha_v/d)$  is a sum of integers, and so  $b$  is well-defined. Define  $\beta_v = d_G(v) - \alpha_v$ . Clearly,  $0 \leq \beta_v < d$ . It is important to observe that  $bd + \sum_{v \in V} \beta_v$  is even since

$$(bd + \sum_{v \in V} \beta_v) \pmod{2} \equiv (\sum_{v \in V} \alpha_v + \sum_{v \in V} \beta_v) \pmod{2} \equiv (\sum_{v \in V} d_G(v)) \pmod{2} \equiv 0 \pmod{2}. \quad (1)$$

The proof of Theorem 1.1 is split into several lemmas. It is convenient to dispose first of the easy case where  $d = 1$ .

**Lemma 3.1** *If  $d = 1$  then  $P(H, G) = \lfloor \frac{\sum_{v \in V} \alpha_v}{2h} \rfloor$ .*

**Proof:** Delete from  $G$  a set of  $b/2$  independent edges ( $b$  is even by (1) since  $\beta_v = 0$  for all  $v \in V$ ). The resulting graph  $G'$  has  $\delta(G') \geq \delta(G) - 1 \geq (1 - \gamma(H))n$ , and satisfies the conditions of Lemma 2.1. Thus,  $G'$  has an  $H$ -decomposition, so

$$P(H, G) \geq (|E| - b/2)/h = \lfloor |E|/h \rfloor = \lfloor \sum_{v \in V} \alpha_v/2h \rfloor.$$

Clearly,  $P(H, G) \leq \lfloor |E|/h \rfloor$  for every graph  $G$ .  $\square$

For the remainder of this section we assume  $d > 1$ . The next lemma establishes an upper bound for  $P(H, G)$ :

**Lemma 3.2**

$$P(H, G) \leq \lfloor \frac{\sum_{v \in V} \alpha_v}{2h} \rfloor.$$

**Proof:** Let  $L$  be an arbitrary  $H$ -packing of  $G$ . Let  $s$  denote the cardinality of  $L$ . Let  $G'$  denote the edge-union of all the members of  $L$ .  $G'$  contains  $sh$  edges. Thus  $G^* = G \setminus G'$  contains  $e(G) - sh$  edges. The degree of each vertex in  $G'$  is  $0 \pmod{d}$  and so the degree of each vertex  $v$  in  $G^*$  is  $d_G(v) \pmod{d}$ . Therefore, the number of edges in  $G^*$  satisfies

$$e(G) - sh = \frac{\sum_{v \in V} \beta_v + cd}{2}$$

for some non-negative integer  $c$ . In particular,  $e(G) \equiv \frac{\sum_{v \in V} \beta_v + cd}{2} \pmod{h}$ . This implies that

$$\frac{2e(G) - \sum_{v \in V} \beta_v}{d} \equiv \frac{\sum_{v \in V} \alpha_v}{d} \equiv c \pmod{\frac{2h}{d}}.$$

Thus, we must have  $c \geq b$ . Therefore,

$$s = \frac{e(G) - \frac{\sum_{v \in V} \beta_v + cd}{2}}{h} \leq \frac{e(G) - \frac{\sum_{v \in V} \beta_v + bd}{2}}{h} = \lfloor \frac{\sum_{v \in V} \alpha_v}{2h} \rfloor.$$

Since  $L$  was an arbitrary  $H$ -packing, we have that  $P(H, G) \leq \lfloor \frac{\sum_{v \in V} \alpha_v}{2h} \rfloor$ .  $\square$

Let  $X = \{v \in V \mid \beta_v > 0\}$ .  $X$  contains all the vertices whose degree in  $G$  is not divisible by  $d$ . Trivially, if  $G$  is nowhere  $d$ -divisible, then  $|X| = n$ . The next lemma supplies a lower bound for  $P(H, G)$  in case  $X \geq n/(10d^3)$ .

**Lemma 3.3** *If  $|X| \geq n/(10d^3)$  then  $P(H, G) \geq \lfloor \sum_{v \in V} \alpha_v / (2h) \rfloor$ .*

**Proof:** We start by choosing an arbitrary set  $B$  of  $b$  vertices of  $X$ . For each  $v \in B$  define  $\nu_v = d + \beta_v$ . For each  $v \in X \setminus B$  define  $\nu_v = \beta_v$ . Our first goal is to show that there exists a spanning subgraph of  $G[X]$ , denoted by  $G^*$ , such that the degree of each vertex  $v$  in  $G^*$  is exactly  $\nu_v$ . This is done by applying Lemma 2.2 to the graph  $G[X]$ . The conditions of the lemma are satisfied since, using the facts that  $|X| \geq n/(10d^3)$ ,  $\epsilon(H) \leq 1/(100d^4)$  and  $n \geq 1000h^5 > 1000d^5$ , we have that

$$\delta(G[X]) \geq |X| - \epsilon(H)n \geq |X| - |X|/(4d + 2) + 2d + 1.$$

Also,  $\nu_v \leq 2d - 1$  and  $\sum_{v \in X} \nu_v = bd + \sum_{v \in V} \beta_v$  is an even number, by (1).

Using  $G^*$  we now consider  $G' = G \setminus E(G^*)$  (i.e.  $G'$  is the spanning subgraph of  $G$  obtained by deleting the edges of  $G^*$ ). If  $v \notin B$ , then the degree of  $v$  in  $G'$  is  $\alpha_v$ . If  $v \in B$  then the degree of  $v$  in  $G'$  is  $\alpha_v - d$ . In any case,  $d \mid \gcd(G')$ , and  $G'$  has  $m$  edges where

$$m = \frac{d}{2} \left( \frac{\sum_{v \in V} \alpha_v}{d} - b \right) \equiv 0 \pmod{h}.$$

Also note that

$$\delta(G') \geq \delta(G) - 2d \geq n(1 - \epsilon(H)) - 2d \geq n(1 - \epsilon(H)) - \frac{\gamma(H)}{2}n = n(1 - \epsilon(H) - \frac{\gamma(H)}{2}) \geq n(1 - \gamma(H)).$$

Thus,  $G'$  satisfies the conditions of Lemma 2.1, and therefore  $G$  has an  $H$ -decomposition. This means that

$$P(H, G) \geq P(H, G') = \frac{m}{h} = \frac{d}{2h} \left( \frac{\sum_{v \in V} \alpha_v}{d} - b \right) = \lfloor \frac{\sum_{v \in V} \alpha_v}{2h} \rfloor.$$

□

By Lemma 3.2 and Lemma 3.3 we have the following corollary:

**Corollary 3.4** *If  $|X| \geq n/(10d^3)$  then  $P(H, G) = \lfloor \frac{\sum_{v \in V} \alpha_v}{2h} \rfloor$ .*

Corollary 3.4 proves Theorem 1.1 in case  $|X| \geq n/(10d^3)$ . It includes the case where  $G$  is nowhere  $d$ -divisible, and shows that even in case  $n/(10d^3) \leq |X| < n$ , there is also a closed formula for  $P(H, G)$ , and, in particular, an  $O(n^2)$  algorithm, since in order to compute the formula, one only needs to know the degree sequence of  $G$ .

For the remainder of this section we may and will assume that  $0 \leq |X| < n/(10d^3)$ . The next lemma shows that in this case, an optimal  $H$ -packing of  $G$  may only leave a small fraction of the edges unpacked.

**Lemma 3.5** *If  $|X| < n/(10d^3)$ , then  $|E| - h \cdot P(H, G) \leq n/(5d^2)$ .*

**Proof:** Consider the subgraph of  $G$  induced by  $V \setminus X$ . This subgraph has minimum degree at least  $n - \epsilon(H)n - |X| \geq (n - |X|)/2$ . Therefore, this subgraph contains a Hamiltonian cycle, and, in particular, a set of  $\lceil n/(20d^3) \rceil$  independent edges. Let  $G^-$  be the subgraph obtained from  $G$  by deleting this set of independent edges. Clearly,  $\delta(G^-) = \delta(G) - 1$ , but  $X^-$ , the set of vertices of  $G^-$  whose degree is not divisible by  $d$  satisfies  $|X^-| = |X| + 2\lceil n/(20d^3) \rceil \geq n/(10d^3)$ . (We have used here the fact that  $d > 1$ , since if  $d = 1$  then, trivially,  $|X| = |X^-| = 0$ .) We may now apply Lemmas 3.2 and 3.3 to  $G^-$ , and obtain by corollary 3.4 that

$$P(H, G^-) = \lfloor \frac{\sum_{v \in V} \alpha_v - 2d\lceil n/(20d^3) \rceil}{2h} \rfloor.$$

Since every packing of  $G^-$  is also a packing of  $G$  we have that

$$\begin{aligned} P(H, G) &\geq P(H, G^-) = \lfloor \frac{2|E| - \sum_{v \in X} \beta_v - 2d\lceil n/(20d^3) \rceil}{2h} \rfloor \geq \\ &\frac{2|E| - \sum_{v \in X} \beta_v - n/(10d^2) - 2d}{2h} - 1. \end{aligned}$$

Thus,

$$|E| - h \cdot P(H, G) \leq \sum_{v \in X} \beta_v/2 + n/(20d^2) + d + h \leq d \frac{n}{20d^3} + \frac{n}{20d^2} + d + h \leq \frac{n}{5d^2}.$$

□

Consider an optimal packing of the edges of  $G$  with copies of  $H$ , and let  $G'$  denote the spanning subgraph of  $G$  consisting of the edges of all the copies of  $H$  in the optimal packing. Put  $G^* = G \setminus G'$ . Obviously, the degree of each vertex  $v$  of  $V$  in  $G^*$  satisfies  $d_{G^*}(v) \equiv \beta_v \pmod{d}$ . Our goal is to determine the number of edges of  $G^*$ . For this purpose we need the following lemma:

**Lemma 3.6** *There exists a subgraph of  $G$ , denoted by  $G^{**}$  which has the same number of edges as  $G^*$ , each vertex  $v \in X$  satisfies  $d_{G^{**}}(v) = \beta_v$ , and each  $v \in V \setminus X$  has  $d_{G^{**}}(v) \in \{0, d\}$ .*

**Proof:** Assume that there exists some vertex  $v \in V$  with  $d_{G^*}(v) > d$ . Let  $D$  be a set of  $d$  neighbors of  $v$  in  $G^*$ . The fact that  $\delta(G) \geq n - \epsilon(H)n$  implies that there are at least  $n - d\epsilon(H)n$  vertices which are neighbors, in  $G$ , of all the vertices of  $D$ . Thus, there is a set  $A \subset V \setminus X$ , with  $|A| \geq n - |X| - d\epsilon(H)n$  where each  $a \in A$  is adjacent to all the vertices of  $D$ . According to Lemma 3.5, the number of edges of  $G^*$  is at most  $n/(5d^2)$ . Thus, there are at most  $0.4n/d^2$  non-isolated vertices in  $G^*$ . Since  $n - |X| - d\epsilon(H)n - 0.4n/d^2 > 0$ , there is some  $a \in A$  which is an isolated vertex of  $G^*$ , with the property that for each  $u \in D$ ,  $(u, a) \in G$ . We may replace each edge  $(v, u) \in G^*$  with the edge  $(u, a)$ , and obtain a subgraph of  $G$ , with the same number of edges of  $G^*$ , each vertex except  $a$  and  $v$  has the same degree in the modified graph as in  $G^*$ , The degree of  $v$  has decreased by  $d$ , and the degree of  $a$  is now exactly  $d$ . By repeating this process as long as there is some vertex with degree larger than  $d$ , we obtain, at the end of this process, the graph  $G^{**}$ .  $\square$

The next four lemmas together supply an algorithm for computing  $P(H, G)$  in case  $|X| < n/(10d^3)$ . The last one also proves the correctness of the algorithm. Let  $k(H, G)$  denote the maximum number of edges in a subgraph  $S$  of  $G[X]$  having the property that each  $v \in X$  has degree at most  $\beta_v$  in  $S$ .

**Lemma 3.7** *There is a polynomial time algorithm which computes  $k(H, G)$ .*

**Proof:** We reduce the problem of computing  $k(H, G)$  to the problem of computing a maximum-weight matching on a graph  $Y$ . We define  $Y$  as follows: For each edge  $e = (a, b)$  with  $a, b \in X$  we create two vertices in  $Y$  which we call  $a_e$  and  $b_e$ , and an edge  $(a_e, b_e)$  connecting them, having weight 1 in  $Y$ . For each vertex  $a \in X$  we create additional  $t = d_{G[X]}(a) - \beta_v$  vertices in  $Y$ , denoted by  $a^1, \dots, a^t$ . We connect each vertex of the form  $a^i$  to each vertex of the form  $a_e$  with an edge whose weight is 2 (thus, the degree of  $a^i$  is  $d_{G[X]}(a)$ , while the degree of  $a_e$  is  $1 + d_{G[X]}(a) - \beta_v$ ). Note that the number of vertices of  $Y$  is  $O(n^2)$ , while the number of edges of  $Y$  is  $O(n^3)$ . We now find a maximum-weight matching in  $Y$ . This can be done in  $O(n^5 \log n)$  time using the algorithm presented in [9].

We claim that if  $M$  is a maximum-weight matching in  $Y$ , then every vertex of the form  $a^i$  is matched. If this were not the case, then all the  $d_{G[X]}(v)$  neighbors of  $a^i$  in  $Y$  are matched. In

particular, there is some edge of the form  $(a_e, b_e)$  which appears in  $M$ . This edge, whose weight is 1, can be deleted from  $M$ , and replaced by the edge  $(a^i, a_e)$  whose weight is 2, contradicting the maximality of  $M$ . It now follows from the construction of  $Y$  and the maximality of  $M$  that the set of edges  $e = (a, b) \in G[X]$  for which  $(a_e, b_e) \in M$ , forms a subgraph  $S$  of  $G[X]$  having the property that each  $v \in X$  has degree at most  $\beta_v$  in  $S$ , and that the number of edges of  $S$  is the maximum possible, subject to these constraints.  $\square$

**Lemma 3.8**  $P(H, G)$  is at most the maximum possible value of

$$\frac{1}{h}(|E| - \frac{1}{2}(b'd + \sum_{v \in X} \beta_v)), \quad (2)$$

subject to the following constraints:

1.  $0 \leq b' \equiv b \pmod{2h/d}$ .
2.  $b'd \geq \sum_{v \in X} \beta_v - 2k'$ , where  $0 \leq k' \leq k(H, G)$ .
3. In case  $1 \leq b' \leq d$  it is also required that  $\sum_{v \in X} \beta_v - 2k' \geq b'(d - b' + 1)$ .

This maximum can be computed in  $O(n)$  time (assuming all values except  $k'$  and  $b'$  are known).

**Proof:** Let  $k'$  denote the number of edges of the graph  $G^{**}$  of Lemma 3.6, in its part induced by the vertices of  $X$ . Clearly,  $0 \leq k' \leq k(H, G)$ . Since each vertex in  $V \setminus X$  has degree divisible by  $d$  in  $G^{**}$ , the sum of the degrees of the vertices of  $V \setminus X$  in  $G^{**}$  is  $b'd$  for some  $b' \geq 0$ . We claim that  $b' \equiv b \pmod{2h/d}$ . This is because the sum of the degrees of  $G^{**}$  or  $G^*$  (it is the same by Lemma 3.6) is  $\sum_{v \in X} \beta_v + b'd$ , the sum of the degrees of  $G'$  is  $2h \cdot P(H, G)$  (since it is an edge disjoint union of an optimal packing) and thus

$$\sum_{v \in X} \beta_v + b'd + 2hP(H, G) = 2|E| = \sum_{v \in V} d_G(v) = \sum_{v \in V} (\alpha_v + \beta_v). \quad (3)$$

Using (3), the definition of  $b$ , and recalling that  $\beta_v = 0$  for  $v \in V \setminus X$  we get that

$$b' + (2h/d)P(H, G) = \sum_{v \in V} \alpha_v/d \equiv b \pmod{2h/d}.$$

It also follows from (3), that

$$P(H, G) = \frac{1}{h}(|E| - \frac{1}{2}(b'd + \sum_{v \in X} \beta_v)).$$

The sum of the degrees in  $G^{**}$  between  $X$  and  $V \setminus X$  is  $\sum_{v \in X} \beta_v - 2k'$ , and therefore we must have that  $b'd \geq \sum_{v \in X} \beta_v - 2k'$ . Finally, consider the case where  $1 \leq b' \leq d$ . There are at most  $b'(b' - 1)/2$

edges of  $G^{**}$  with both endpoints in  $V \setminus X$ . Thus, there are at least  $b'd - b'(b' - 1) = b'(d - b' + 1)$  edges of  $G^{**}$  between  $X$  and  $V \setminus X$ . Hence we must have  $\sum_{v \in X} \beta_v - 2k' \geq b'(d - b' + 1)$ .

We have proved that there exist  $b'$  and  $k'$  satisfying the constraints of the lemma, such that  $P(H, G) = \frac{1}{h}(|E| - \frac{1}{2}(b'd + \sum_{v \in X} \beta_v))$ . Hence, by trying all the possible combinations of  $b'$  and  $k'$  satisfying the constraints, we have that  $P(H, G)$  is at most the maximum possible value of (2) subject to the constraints. Computing this maximum can be done in  $O(n)$  time since  $0 \leq k' \leq k(H, G) = O(n)$  and for every possible value of  $k'$  in this range, the minimum possible value of  $b'$  satisfying the constraints can be found in constant time.  $\square$

Our next goal is to show that the upper bound for  $P(H, G)$  computed in Lemma 3.8, is, in fact, the exact value of  $P(H, G)$ , or at most one greater than the exact value, and we can determine which of these two options holds in polynomial time.

**Lemma 3.9** *Let  $k'$  and  $b'$  give the maximum to (2), subject to the constraints of Lemma 3.8. Then:*

- *There exists a subgraph  $G^*$  of  $G$  in which every  $v \in X$  has degree  $\beta_v$ , exactly  $b'$  vertices of  $V \setminus X$  have degree  $d$ , and there are  $k'$  edges in the subgraph of  $G^*$  induced by  $X$ .*

**or else** *the following must hold:*

- *There exists a subgraph  $G^*$  of  $G$  in which every  $v \in X$  has degree  $\beta_v$ , exactly  $b' + 2h/d$  vertices of  $V \setminus X$  have degree  $d$ , and there are  $k'$  edges in the subgraph of  $G^*$  induced by  $X$ .*

*If  $b' \geq d - 1$  or  $b' = 0$  then the first case always holds, and if  $1 \leq b' \leq d - 2$  then it can be verified in polynomial time whether the first case holds.*

**Proof:** Our first goal is to construct a subgraph  $S$  of  $G$ , on the vertices of  $X$ , which satisfies the following three requirements:

1.  $S$  has  $k'$  edges.
2. Each  $v \in X$  has degree at most  $\beta_v$  in  $S$ .
3.  $\beta_v - d_S(v) \leq b'$  for each  $v \in X$ .

Clearly, the existence of  $S$  is a necessary condition if we wish for the first case in the lemma to hold. Using Lemma 3.7 and the fact that  $0 \leq k' \leq k(H, G)$  we know that there exists graphs which satisfy the first two requirements. Recall that  $\beta_v \leq d - 1$ . Thus, if  $b' \geq d - 1$ , then the third requirement is nil, so  $S$  exists. If  $b' = 0$  then we know from the second constraint in Lemma 3.8 that  $\beta_v = d_S(v)$  for each graph which satisfies the first two requirements, so, once again, the third requirement is nil, so  $S$  exists. However, if  $1 \leq b' \leq d - 2$ , the third requirement is not nil. We can still, however, determine if  $S$  exists in polynomial time. This is done as follows: Let

$s = \sum_{v \in X} \beta_v - 2k'$ . According to the constraints in Lemma 3.8, and since  $b' \leq d - 2$ , we have

$$d(d - 2) \geq b'd \geq \sum_{v \in X} \beta_v - 2k' = s.$$

Thus,  $s$  is bounded by a constant. Thus, there are only at most  $|X|^{d(d-2)}$  possible degree sequences for  $S$ . We will try every possible degree sequence, and for each degree sequence,  $\{S(v) \mid v \in X\}$  we can determine if  $S$  exists using the algorithm similar to the one in Lemma 3.7 (the difference is that instead of requiring that each vertex have degree at most  $\beta_v$  as in Lemma 3.7, we now require that each vertex have degree exactly  $S(v)$ ). Clearly, the same weighted-matching algorithm solves this problem). If at least one degree sequence is satisfied, then  $S$  exists. Otherwise,  $S$  does not exist, so the first case in the lemma cannot hold. Note that the overall running time for detecting the existence of  $S$  is  $O(n^{d(d-2)} \cdot n^5 \log n)$ , which is polynomial. In case  $S$  does not exist we can still create a graph  $S$  which only satisfies the first two requirements.

For  $v \in X$ , define  $\gamma_v = \beta_v - d_S(v)$ , and define  $s = \sum_{v \in X} \gamma_v = \sum_{v \in X} \beta_v - 2k'$ . The graph  $S$  will be the subgraph of  $G^*$  induced by  $X$ . It remains to define the other edges of  $G^*$ . Let  $Z = \{z_1, \dots, z_t\}$  be a set of new vertices. If  $S$  satisfies all three requirements then  $t = b'$ . If  $S$  satisfies only the first two requirements then  $t = b' + 2h/d$  (if  $b' = 0$  then  $Z = \emptyset$ ). Note that, in any case,  $t \geq \gamma_v$  for each  $v \in V$ . This is because  $d = \gcd(H)$  so  $H$  has at least  $d(d+1)/2$  edges, and therefore  $h \geq d(d+1)/2$  so  $2h/d \geq d+1 > \beta_v \geq \gamma_v$ .

Let  $\{v_1, \dots, v_{|X|}\}$  be an ordering of  $X$ . For  $i = 1, \dots, |X|$  we perform the following process which assigns edges between  $Z$  and  $X$ : The process assigns  $\gamma_{v_i}$  edges between  $v_i$  and  $Z$  in such a way that after the assignment, the degrees of each pair of vertices of  $Z$  differ by at most 1. This can clearly be done since  $\gamma_{v_i} \leq t$ . After the process ends, consider the graph  $T$  on the vertices  $X \cup Z$  obtained by the union of  $S$  and the edges assigned between  $X$  and  $Z$ .  $T$  clearly satisfies the following properties:

1. Each  $v \in X$  has degree  $\beta_v$  in  $T$ .
2. If  $t$  divides  $s$  then each  $z_i$  has degree  $s/t$  in  $T$ . Otherwise, Exactly  $s \bmod t$  vertices of  $Z$  have degree  $\lceil s/t \rceil$  and the other  $t - (s \bmod t)$  vertices of  $Z$  have degree  $\lfloor s/t \rfloor$ .

Recall the second constraint in Lemma 3.8, which states that  $s \leq b'd$ . Thus,  $s \leq td$  and so  $s/t \leq d$ , and therefore no vertex of  $Z$  has degree greater than  $d$  in  $T$ .

Our next goal is to add to  $T$  edges between vertices of  $Z$  so that after this addition, the degree of each vertex of  $Z$  will be exactly  $d$ . The sum of the degrees of the vertices of  $Z$  should therefore be  $td$  after the addition, while the sum of the degrees of the vertices of  $Z$  in  $T$  is  $s$  prior to the addition. Thus, it suffices to show that there exists a graph  $R$  on  $t$  vertices, with  $(td - s)/2$  edges, such that the degrees of each two vertices of  $R$  differ by at most one. In fact, in order to show that

$R$  exists we only need to show that  $(td - s)/2$  is an integer and that  $t(t - 1)/2 \geq (td - s)/2$  since it is a well-known fact that the complete graph on  $t$  vertices contains a graph with  $t'$  edges for every  $0 \leq t' \leq t(t - 1)/2$  where the degrees of any two vertices differ by at most 1. The fact that  $(td - s)/2$  is an integer follows from (1), from the fact that  $t \equiv b \pmod{2h/d}$ , and from the definition of  $s$  which implies that  $s \equiv \sum_{v \in V} \beta_v \pmod{2}$ . The fact that  $t(t - 1)/2 \geq (td - s)/2$  is proved as follows. If  $t \geq d + 1$  then clearly  $t(t - 1)/2 \geq (td - s)/2$ . If  $t \leq d$  then also  $b' \leq d$  (in fact,  $t = b'$  in this case). We may therefore use constraint 3 in Lemma 3.8 which states that  $s \geq b'(d - b' + 1)$  and thus

$$t(t - 1)/2 = b'(b' - 1)/2 = (b'd - b'(d - b' + 1))/2 \geq (b'd - s)/2 = (td - s)/2.$$

After adding to  $T$  the required set of edges as described in the previous paragraph, we obtain a graph  $T'$  on the vertices  $X \cup Z$ , such that  $d_{T'}(v) = \beta_v$  for  $v \in X$  and  $d_{T'}(z) = d$  for  $z \in Z$ . Our goal is to embed  $T'$  into  $G$ , and this embedding clearly constitutes the graph  $G^*$  in the statement of the Lemma. We need to show that such an embedding can be done. Namely, we must assign each  $z_i \in Z$  to a vertex  $u_i \in V \setminus X$ , such that if  $(z_i, v) \in T'$  where  $v \in X$ , then  $(u_i, v) \in G$ , and if  $(z_i, z_j) \in T'$  where  $j < i$  then  $(u_i, u_j) \in G$ . We will perform this assignment sequentially beginning with  $z_1$ . Assume that we have already mapped  $z_1, \dots, z_i$  to  $u_1, \dots, u_i$  respectively. We need to show how  $z_{i+1}$  is mapped. We know exactly which neighbors  $u_{i+1}$  must have. This set of neighbors contains at most  $d$  elements. Since  $\delta(G) \geq n - \epsilon(H)n$ , there are at least  $n - d\epsilon(H)n - |X| - i > 0$  candidates for the role of  $u_i$ , so we pick one of them.  $\square$

The following lemma completes the description of the algorithm, and proves its correctness:

**Lemma 3.10** *Let  $b'$  maximize (2) subject to the constraints of Lemma 3.8. If the first case of Lemma 3.9 holds for some  $k'$  which satisfies the constraints together with  $b'$ , then*

$$P(H, G) = \frac{1}{h}(|E| - \frac{1}{2}(b'd + \sum_{v \in X} \beta_v)).$$

*Otherwise*

$$P(H, G) = \frac{1}{h}(|E| - \frac{1}{2}(b'd + \sum_{v \in X} \beta_v)) - 1.$$

*We can verify whether there exists a  $k'$  satisfying the first case of Lemma 3.9 in polynomial time. Consequently,  $P(H, G)$  can be computed in polynomial time.*

**Proof:** By Lemma 3.8, we can compute a pair  $(k', b')$  which maximizes (2) in  $O(n)$  time. Any other pair which maximizes (2) has the same  $b'$ , but may have a different  $k'$ . As shown in the proof of Lemma 3.8, all valid values of  $k'$  achieving the maximum can be computed in  $O(n)$  time.

Consider first the case that for some maximizing pair  $(k', b')$ , the first case of Lemma 3.9 holds. In this case,  $G^*$  is a subgraph of  $G$  having  $(b'd + \sum_{v \in X} \beta_v)/2$  edges, and having  $\delta(G^*) \leq d$ . Thus,

$G' = G \setminus G^*$  has exactly  $|E| - (b'd + \sum_{v \in X} \beta_v)/2$  edges, and  $\delta(G') \geq \delta(G) - d \geq n(1 - \gamma(H))$ . The other divisibility conditions of Lemma 2.1 are also clearly satisfied by  $G'$ , so  $G'$  has an  $H$ -decomposition. Thus,

$$P(H, G) \geq P(H, G') = \frac{1}{h}(|E| - \frac{1}{2}(b'd + \sum_{v \in X} \beta_v)).$$

Since, by Lemma 3.8,  $P(H, G)$  cannot exceed (2), the last inequality is an equality.

Next, consider the case that for every maximizing pair  $(k', b')$  the first case of Lemma 3.9 does not hold. This means that  $P(H, G)$  cannot reach the value of (2), since if it did, we would have, by Lemma 3.6, a graph  $G^{**}$  which does satisfy the first case of Lemma 3.9, a contradiction. Therefore,

$$P(H, G) \leq \frac{1}{h}(|E| - \frac{1}{2}(b'd + \sum_{v \in X} \beta_v)) - 1.$$

According to Lemma 3.9, there exists a subgraph  $G^*$  of  $G$  with  $((b' + 2h/d)d + \sum_{v \in X} \beta_v)/2$  edges, and having  $\delta(G^*) \leq d$ . Thus,  $G' = G \setminus G^*$  has exactly  $|E| - ((b' + 2h/d)d + \sum_{v \in X} \beta_v)/2$  edges, and  $\delta(G') \geq \delta(G) - d \geq n(1 - \gamma(H))$ . As in the previous case,  $G'$  has an  $H$ -decomposition by Lemma 2.1. Thus,

$$P(H, G) \geq P(H, G') = \frac{1}{h}(|E| - \frac{1}{2}((b' + 2h/d)d + \sum_{v \in X} \beta_v)) = \frac{1}{h}(|E| - \frac{1}{2}(b'd + \sum_{v \in X} \beta_v)) - 1.$$

Since the upper and lower bounds for  $P(H, G)$  coincide, the last inequality is an equality.

By Lemma 3.9, we can verify in polynomial time if a maximizing pair  $(k', b')$  satisfies the first case of Lemma 3.9 or not. As all valid values of  $k'$  can be computed in  $O(n)$  time, we can determine in polynomial time if some maximizing pair satisfies the first case of Lemma 3.9. Thus, by computing  $k(H, G)$  in  $O(n^5 \log n)$  time (Lemma 3.7), and then computing all maximizing pairs  $(k', b')$  in  $O(n)$  time (Lemma 3.8) and applying Lemma 3.9 to each of them (if  $b' \geq d - 1$  or  $b' = 0$  we do not even need to apply the algorithmic part of Lemma 3.9 since we are guaranteed that the first case of Lemma 3.9 holds), we get by the current Lemma that  $P(H, G)$  can be computed in polynomial time  $\square$

In case the graph  $G$  is  $d$ -divisible, Lemma 3.8 already establishes a closed formula for  $P(H, G)$ . Since in this case,  $X = \emptyset$ , and  $k(H, G) = 0$ , Lemma 3.8 states that  $P(H, G)$  is at most the maximum possible value of  $|E|/h - b'd/(2h)$  subject to  $b' \equiv b \pmod{2h/d}$  and  $b' \notin \{1, \dots, d\}$ . Lemmas 3.9 and 3.10 show that for these values of  $b'$ ,  $P(H, G)$  is, in fact, *exactly* the maximum possible value of  $|E|/h - b'd/(2h)$  subject to the above conditions on  $b'$ . Solving this, we get that  $b' = b$  if  $b \notin \{1, \dots, d\}$  and  $b' = b + 2h/d$  if  $b \in \{1, \dots, d\}$ . In the first case

$$P(H, G) = |E|/h - bd/(2h) = \lfloor |E|/h \rfloor = \lfloor \frac{\sum_{v \in V} \alpha_v}{2h} \rfloor.$$

In the second case,

$$P(H, G) = |E|/h - (b + 2h/d)d/(2h) = |E|/h - bd/(2h) - 1 = \lfloor |E|/h \rfloor - 1 = \lfloor \frac{\sum_{v \in V} \alpha_v}{2h} \rfloor - 1.$$

Note that, by the definition of  $b$ ,  $1 \leq b \leq d$  occurs if and only if  $2|E|/d \bmod (2h/d)$  is in the range  $1, \dots, d$ , which is equivalent to saying that  $0 < |E| \bmod h \leq d^2/2$ .  $\square$

## 4 Covering dense graphs

In this section we prove Theorem 1.2. Before we start with the proof, we need several definitions and a lemma. Recall that a *multigraph* is a graph in which multiple edges and loops are allowed. During the rest of this section, all multigraphs considered are assumed to have no loops. The degree of a vertex  $v$  in a multigraph is defined as the number of edges incident with  $v$ , taking multiplicity into account (i.e. an edge with multiplicity  $k$  contributes  $k$  to the degrees of its incident vertices). For a multigraph  $M$ , let  $u(M)$  denote the *underlying graph*, where every edge only has multiplicity one. The next lemma is crucial to our proof of Theorem 1.2.

**Lemma 4.1** *Let  $H$  be a graph with  $h \geq 2$  edges, and no isolated vertices. Then, if  $G$  is an  $n$ -vertex graph with  $\delta(G) \geq (1 - 1/(20h^3))n$ , and  $G^*$  is an  $n$ -vertex multigraph, with  $\Delta(G^*) \leq h$ , such that  $u(G^*)$  is a subgraph of  $G$ , then there exists an  $n$ -vertex multigraph  $R$  with the following properties:*

1.  $G^*$  is a sub-multigraph of  $R$ , and  $u(R)$  is a subgraph of  $G$ . (One can imagine this as  $u(R)$  being a sandwich between  $G$  and  $u(G^*)$ ).
2.  $R \setminus G^*$  is a graph (i.e. the edges of  $R$  not belonging to  $G^*$  have multiplicity one, or, in other words,  $E(R \setminus G^*) = E(u(R) \setminus u(G^*))$ ).
3.  $\Delta(R) \leq 4h^2$ .
4.  $R$  has an  $H$ -decomposition.

**Proof:** We shall prove the lemma by induction on  $e(G^*)$ , the number of edges of  $G^*$ . In fact, we will show that if  $e(G^*) = k$ , then one may construct  $R$ , having the properties guaranteed by the lemma, with the additional requirements that

$$e(R) \leq kh, \tag{4}$$

and that for every vertex  $v$ ,

$$d_R(v) \leq h \cdot d_{G^*}(v) + 3h^2 \leq h^2 + 3h^2 \leq 4h^2. \tag{5}$$

The basis of the induction,  $k = 0$ , holds since in this case  $R = G^*$  is the empty graph on  $n$  isolated vertices, and all properties trivially hold. Now suppose  $e(G^*) = k + 1$ . Put  $G^k = G^* \setminus \{(a, b)\}$  where  $(a, b)$  is an arbitrary edge of  $G^*$ . Since  $e(G^k) = k$ , we have, according to the induction hypothesis, that there exists a multigraph  $R^k$ , with all the above properties, with respect to  $G^k$  and  $G$ .

If  $(a, b) \in R^k$ , we may take  $R = R^k$ , and we are done. Assume, therefore, that  $(a, b) \notin R^k$ . Since  $e(R^k) \leq kh$ , and since  $k = e(G^k) \leq nh/2$  we have  $e(R^k) \leq nh^2/2$ . Thus, there are at least  $n/2$  vertices with degree at most  $2h^2$  in  $R^k$ . Since  $\Delta(R^k) \leq 4h^2$  we have, therefore, that there is a set of vertices  $X$ , with  $|X| \geq n/2 - 8h^2 - 2$ , such that for every  $v \in X$ ,  $d_{R^k}(v) \leq 2h^2$ ,  $v \neq a$ ,  $v \neq b$ ,  $(v, a) \notin R^k$  and  $(v, b) \notin R^k$ . We can find in  $X$  a large subset  $X'$  with the additional property that every  $v \in X'$  is connected to both  $a$  and  $b$  in  $G$ . Since  $\delta(G) \geq n(1 - 1/(20h^3))$  there are at most  $n/(10h^3)$  vertices in  $G$  which are not connected to either  $a$  or  $b$ . Thus the desired  $X'$  contains  $|X'| \geq |X| - n/(10h^3)$  vertices. We claim that there is an independent set  $T \subset X'$  in  $R^k$  containing  $t = |T|$  vertices, where  $|T| + 2$  is the number of vertices of  $H$ , and furthermore,  $T$  induces a *complete graph* in  $G$ . Note first that  $G[X']$ , has high minimum degree. Indeed, for  $v \in X'$ ,

$$d_{G[X']}(v) \geq |X'| - \frac{n}{20h^3} \geq |X'| \left(1 - \frac{1}{4h^3}\right)$$

where the last inequality uses the fact that  $|X'| \geq n/5$ . This, in turn, is true since

$$|X'| \geq |X| - n/(10h^3) \geq n/2 - 8h^2 - 2 - n/(10h^3) \geq n/5$$

which follows from the facts that  $n \geq 20h^3$ , and that  $h \geq 2$ . According to Turán's Theorem,  $G[X']$  contains a complete graph on a set  $Y$  of  $4h^3$  vertices. It now suffices to show that  $R^k[Y]$  contains the required independent subset  $T$ . Since  $H$  has no isolated vertices, it has at most  $2h - 2$  vertices, and thus it suffices to show that  $R^k[Y]$  has an independent set of size  $2h - 2$ . Since  $\Delta(R^k[Y]) \leq \Delta(R^k[X]) \leq 2h^2$ , it is enough to show that  $|Y|/(2h^2 + 1) \geq 2h - 2$ . Indeed, this holds since  $|Y| = 4h^3$ . We have proved that the required set  $T$  exists. Note that the definitions of  $X$  and  $X'$  imply that  $Z = T \cup \{a, b\}$  is an independent set of  $R^k$ , with exactly the same cardinality as the vertex-set of  $H$ , and  $G[Z]$  is a complete graph. We can now arbitrarily embed a copy of  $H$  on the vertex set  $Z$ , such that  $(a, b)$  is an edge of this copy. Let  $F$  denote the set of edges of this copy. Clearly,  $|F| = h$  and  $(a, b) \in F$ . Put  $R = R^k \cup F$ . Our construction shows that:

1.  $G^*$  is a sub-multigraph of  $R$ . (since  $G^k$  is a spanning sub-multigraph of  $R^k$ , and since  $(a, b) \in R$ ).
2.  $u(R)$  is a spanning subgraph of  $G$ , since  $u(R^k)$  is a spanning subgraph of  $G$ , and  $u(R)$  is obtained from  $u(R^k)$  by adding a copy of  $H$  containing only edges of  $G$ , since these edges belong to  $G[Z]$ .

3.  $E(R \setminus G^*) = E(R^k \setminus G^k) \cup (F \setminus \{a, b\})$ . This is an edge-disjoint union of two graphs, and therefore  $R \setminus G^*$  is a *graph*, as required.
4. If  $v \notin Z$  then  $d_R(v) = d_{R^k}(v) \leq h \cdot d_{G^k}(v) + 3h^2 \leq h \cdot d_{G^*}(v) + 3h^2$ . If  $v \in T$  then  $d_R(v) \leq d_{R^k}(v) + h \leq 2h^2 + h \leq h \cdot d_{G^*}(v) + 3h^2$ . Finally, if  $v \in \{a, b\}$  then  $d_R(v) \leq d_{R^k}(v) + h \leq h \cdot d_{G^k}(v) + 3h^2 + h = h \cdot d_{G^*}(v) + 3h^2$ . In any case, we have shown that  $d_R(v) \leq h \cdot d_{G^*}(v) + 3h^2$  for every vertex  $v$ , as required by (5).
5.  $R$  has an  $H$ -decomposition since  $R^k$  has an  $H$ -decomposition and since  $R = R^k \cup F$  where  $F$  is a copy of  $H$ , and no edge of  $F$  appears in  $R^k$ .
6.  $e(R) = e(R^k) + h \leq kh + h = (k + 1)h$ , as required by (4).

This completes the induction step, and hence the proof.  $\square$

**Proof of Theorem 1.2** Given  $H$ , we choose

$$N(H) = \max\{N_0(H), \frac{8h^2}{\gamma(H)}, 1000h^5\}$$

where  $N_0(H)$  and  $\gamma(H)$  are as in Lemma 2.1. We also choose

$$\epsilon(H) = \min\{\frac{\gamma(H)}{2}, \frac{1}{20h^3}, \frac{1}{100d^4}\}.$$

Now let  $G$  be a graph with  $n > N(H)$  vertices and  $\delta(G) > (1 - \epsilon(H))n$ . We need to show how  $C(H, G)$  can be computed in  $O(n^{2.5})$  time, and, moreover, supply a closed formula for  $C(H, G)$  in case  $G$  is either  $d$ -divisible or nowhere  $d$ -divisible. There are many similarities between the proof of the packing result in Section 3, and the proof here. Whenever we encounter such similarities, we shall only note them, instead of reproving them.

Let  $0 \leq b < 2h/d$  satisfy  $(\sum_{v \in V} \alpha_v)/d \equiv -b \pmod{2h/d}$  (note the minus sign). As in Section 3, since  $2h/d$  is always an integer, and since  $\sum_{v \in V} (\alpha_v/d)$  is a sum of integers, we have that  $b$  is well-defined. Define  $\beta_v = \alpha_v - d_G(v)$ . (recall that  $\alpha_v$ , in the covering case, is the degree of  $v$  rounded *up* to the closest multiple of  $d$ ). Note that  $0 \leq \beta_v < d$ . A similar reasoning to the one shown in Section 3 asserts that  $bd + \sum_{v \in V} \beta_v$  is even. The proof of Theorem 1.2 is split into several lemmas. We begin by proving a lower bound for  $C(H, G)$ .

**Lemma 4.2**

$$C(H, G) \geq \lceil \frac{\sum_{v \in V} \alpha_v}{2h} \rceil.$$

**Proof:** Let  $L$  be an arbitrary  $H$ -covering of  $G$ . Let  $s$  denote the cardinality of  $L$ . Let  $G'$  be the  $n$ -vertex multigraph obtained by the edge-union of all the members of  $L$ . That is, an edge of  $G'$

has multiplicity  $k$  if it appears in  $k$  members of  $L$ . Clearly,  $G'$  contains  $sh$  edges. Since  $G$  is a spanning *subgraph* of  $G'$  (in fact,  $G = u(G')$ ), we may define the multigraph  $G^* = G' \setminus G$ .  $G^*$  contains  $sh - e(G)$  edges. The degree of every vertex in  $G'$  is  $0 \pmod d$  and so the degree of every vertex  $v$  in  $G^*$  is  $(-d_G(v)) \pmod d$ . Therefore, the number of edges in  $G^*$  satisfies

$$sh - e(G) = \frac{cd + \sum_{v \in V} \beta_v}{2}$$

for some non-negative integer  $c$ . In particular,  $e(G) = (-\frac{\sum_{v \in V} \beta_v + cd}{2}) \pmod h$ . This implies that

$$\frac{2e(G) + \sum_{v \in V} \beta_v}{d} = \frac{\sum_{v \in V} \alpha_v}{d} = (-c) \pmod{(2h/d)}.$$

Thus, we must have  $c \geq b$ . Therefore,

$$s = \frac{e(G) + \frac{cd + \sum_{v \in V} \beta_v}{2}}{h} \geq \frac{e(G) + \frac{bd + \sum_{v \in V} \beta_v}{2}}{h} = \lceil \frac{\sum_{v \in V} \alpha_v}{2h} \rceil.$$

Since  $L$  was an arbitrary  $H$ -covering, we have  $C(H, G) \geq \lceil \frac{\sum_{v \in V} \alpha_v}{2h} \rceil$ .  $\square$

Let  $X = \{v \in V \mid \beta_v > 0\}$ .  $X$  contains all the vertices whose degrees in  $G$  are not divisible by  $d$ . Trivially, if  $G$  is nowhere  $d$ -divisible, then  $|X| = n$ . The next lemma supplies an upper bound for  $C(H, G)$  in case  $X \geq n/(10d^3)$ .

**Lemma 4.3** *If  $|X| \geq n/(10d^3)$  then  $C(H, G) \leq \lceil \sum_{v \in V} \alpha_v / (2h) \rceil$ . Furthermore, there exists an  $H$ -covering of  $G$  which obtains this upper bound, in which every edge of  $G$  is covered at most twice.*

**Proof:** We start by choosing an arbitrary set  $B$  of  $b$  vertices of  $X$ . For each  $v \in B$  define  $\nu_v = d + \beta_v$ . For each  $v \in X \setminus B$  define  $\nu_v = \beta_v$ . Exactly as in the proof of Lemma 3.3, we know that there exists a spanning subgraph of  $G[X]$ , denoted by  $G^*$ , such that the degree of each vertex  $v$  in  $G^*$  is exactly  $\nu_v$  (recall that this is done by applying Lemma 2.2 to the graph  $G[X]$ ). We shall consider  $G^*$  as an  $n$ -vertex subgraph of  $G$  by adding to  $G^*$   $n - |X|$  isolated vertices.

We now wish to apply Lemma 4.1 to  $G^*$ . (Although Lemma 4.1 assumes that  $G^*$  is a multigraph, we only use here the special case where  $G^*$  is a graph). This can be done since  $\Delta(G^*) \leq 2d - 1 \leq d(d+1)/2 \leq h$ , since  $\delta(G) \geq (1 - 1/(20h^3))n$ , and since  $G^*$  is a subgraph of  $G$ . According to Lemma 4.1, there exists a spanning subgraph of  $G$ , denoted by  $R$ , which contains  $G^*$ ,  $\delta(R) \leq 4h^2$ , and  $R$  has an  $H$ -decomposition. Let  $G'$  be the spanning subgraph of  $G$  which is obtained by deleting from  $G$  the edges of  $R$  which are not in  $G^*$ . We claim that  $d \mid \gcd(G')$ . To see this, note that the fact that  $R$  has an  $H$ -decomposition implies that  $d \mid \gcd(R)$ . Since the degree of each vertex  $v$  of  $G^*$  is  $\beta_v \pmod d$ , it follows that the degree of  $v$  in  $R \setminus G^*$  is  $(-\beta_v) \pmod d$ . Since the degree of  $v$  in

$G$  is also  $(-\beta_v) \pmod d$ , it follows that the degree of  $v$  in  $G'$  is  $0 \pmod d$ . Now we claim that  $e(G')$  is  $0 \pmod h$ . This is because  $e(R) = 0 \pmod h$ , and since, using the definition of  $b$ , we have

$$e(G') = e(G) - e(R) + e(G^*) = e(G) - e(R) + \frac{bd + \sum_{v \in X} \beta_v}{2} = \frac{d}{2} \left( \frac{\sum_{v \in V} \alpha_v}{d} + b \right) - e(R) \equiv 0 \pmod h.$$

Also note that

$$\delta(G') \geq \delta(G) - 4h^2 \geq (1 - \epsilon(H))n - 4h^2 \geq (1 - \gamma(H))n,$$

where the last inequality follows from the facts that  $n \geq 8h^2/\gamma(H)$  and  $\epsilon(H) \leq \gamma(H)/2$ . Since, also,  $n > N_0(H)$ , we have that  $G'$  satisfies the conditions of Lemma 2.1, and therefore  $G'$  has an  $H$ -decomposition. The union of the  $H$ -decomposition of  $G'$  and the  $H$ -decomposition of  $R$  yields a covering of  $G$  in which all the edges of  $G$ , but the edges of  $G^*$ , are covered once. The edges of  $G^*$  are covered twice. The overall number of copies of  $H$  in both decompositions is, therefore, exactly  $(e(G) + e(G^*))/h$ . Thus,

$$C(H, G) \leq \frac{e(G) + e(G^*)}{h} = \frac{e(G) + (bd + \sum_{v \in X} \beta_v)/2}{h} = \frac{d}{2h} \left( \frac{\sum_{v \in V} \alpha_v}{d} + b \right) = \lceil \frac{\sum_{v \in V} \alpha_v}{2h} \rceil.$$

□

By Lemma 4.2 and Lemma 4.3 we have the following corollary:

**Corollary 4.4** *If  $|X| \geq n/(10d^3)$  then  $C(H, G) = \lceil \frac{\sum_{v \in V} \alpha_v}{2h} \rceil$ . Furthermore, there exists an optimal covering in which every edge is covered at most twice.*

Corollary 4.4 proves Theorem 1.2 in case  $|X| \geq n/(10d^3)$ . It includes the case where  $G$  is nowhere  $d$ -divisible, and shows that even in case  $n/(10d^3) \leq |X| < n$ , there is also a closed formula for  $C(H, G)$ , and, in particular, an  $O(n^2)$  algorithm, since in order to compute the formula, one only needs to know the degree sequence of  $G$ .

For the remainder of this section we may and will assume that  $0 \leq |X| < n/(10d^3)$ . Consider an optimal covering of the edges of  $G$  with copies of  $H$ , and let  $G'$  denote the multigraph obtained by the union of all the copies of  $H$  in the optimal covering. Clearly,  $G$  is a spanning subgraph of  $G'$ . Put  $G^* = G' \setminus G$ . Note that  $G^*$  may be a multigraph since there may be edges covered more than twice in the optimal covering. Obviously, the degree of each vertex  $v$  of  $V$  in  $G^*$  satisfies  $d_{G^*}(v) \equiv \beta_v \pmod d$ . Our goal is to determine the number of edges of  $G^*$ . For this purpose we need the following lemma:

**Lemma 4.5** *There exists a sub-multigraph of  $G'$ , denoted by  $G^{**}$  which has the same number of edges as  $G^*$ , each vertex  $v \in X$  satisfies  $d_{G^{**}}(v) = \beta_v$ , and each  $v \in V \setminus X$  has  $d_{G^{**}}(v) \equiv 0 \pmod d$ .*

**Proof:** The proof is an analog to the proof of Lemma 3.6 in Section 3. In fact, it is simpler, since we allow multiple edges in  $G^{**}$ , and we allow vertices of  $V \setminus X$  to have degrees larger than  $d$  (as long as they are multiples of  $d$ ). Thus, we do not need any sparsity requirements placed on  $G^*$ . Hence we do not need an equivalent of Lemma 3.5 as we did in the proof of Lemma 3.6 (although an equivalent of Lemma 3.5 does hold for the covering case as well). Considering these relaxations, the details of the proof can be found in Lemma 3.6.  $\square$

The next three lemmas together supply an algorithm for computing  $C(H, G)$  in case  $|X| < n/(10d^3)$ . The last one also proves the correctness of the algorithm. Let  $k(H, G)$  denote the maximum number of edges in a multigraph  $S$  on the vertices of  $X$ , where each edge of  $S$  is a copy of an edge  $G$ , and each  $v \in X$  has degree at most  $\beta_v$  in  $S$ .

**Lemma 4.6** *There is an algorithm whose running time is  $O(n^{2.5})$ , which computes  $k(H, G)$ .*

**Proof:** We reduce the problem of computing  $k(H, G)$  to the problem of computing a maximum matching on a graph  $Y$ . We define  $Y$  as follows: For each  $v \in X$ , create  $\beta_v$  copies of  $v$  in  $Y$ , and for each  $(u, v) \in G[X]$ , connect every copy of  $v$  in  $Y$  to every copy of  $u$  in  $Y$ . Clearly, a maximum matching in  $Y$  is equivalent to a multigraph  $S$  on the vertices of  $X$  with the maximum possible number of edges, satisfying the required constraints. Note that  $Y$  has  $\sum_{v \in X} \beta_v = O(n)$  vertices, so we can compute  $k(H, G)$  in  $O(n^{2.5})$  time using the algorithm of Even and Kariv [8].  $\square$

**Lemma 4.7**  *$C(H, G)$  is at least the minimum possible value of*

$$\frac{1}{h}(|E| + \frac{1}{2}(b'd + \sum_{v \in X} \beta_v)), \quad (6)$$

*subject to the following constraints:*

1.  $0 \leq b' \equiv b \pmod{2h/d}$ .
2.  $b'd \geq \sum_{v \in X} \beta_v - 2k'$ , where  $0 \leq k' \leq k(H, G)$ .
3. In case  $b' = 1$  it is also required that  $\sum_{v \in X} \beta_v - 2k' \geq d$ .

*This minimum can be computed in  $O(n)$  time (assuming all values except  $k'$  and  $b'$  are known).*

**Proof:** Let  $k'$  denote the number of edges of the multigraph  $G^{**}$  of Lemma 4.5, in its part induced by the vertices of  $X$ . Clearly,  $0 \leq k' \leq k(H, G)$ . Since each vertex in  $V \setminus X$  has degree divisible by  $d$  in  $G^{**}$ , the sum of the degrees of the vertices of  $V \setminus X$  in  $G^{**}$  is  $b'd$  for some  $b' \geq 0$ . We claim that  $b' \equiv b \pmod{2h/d}$ . This is because the sum of the degrees of  $G^{**}$  or  $G^*$  (it is the same by

Lemma 4.5) is  $\sum_{v \in X} \beta_v + b'd$ , the sum of the degrees of  $G'$  is  $2h \cdot C(H, G)$  (since it is a union of the members of an optimal covering) and thus

$$2hC(H, G) - \sum_{v \in X} \beta_v - b'd = 2|E| = \sum_{v \in V} d_G(v) = \sum_{v \in V} (\alpha_v - \beta_v). \quad (7)$$

Using (7), the definition of  $b$ , and recalling that  $\beta_v = 0$  for  $v \in V \setminus X$  we get that

$$(2h/d)C(H, G) - b' = \sum_{v \in V} \alpha_v/d \equiv -b \pmod{(2h/d)}.$$

It also follows from (7), that

$$C(H, G) = \frac{1}{h}(|E| + \frac{1}{2}(b'd + \sum_{v \in X} \beta_v)).$$

The sum of the degrees in  $G^{**}$  between  $X$  and  $V \setminus X$  is  $\sum_{v \in X} \beta_v - 2k'$ , and therefore we must have that  $b'd \geq \sum_{v \in X} \beta_v - 2k'$ . Finally, consider the case where  $b' = 1$ . In this case, there are no edges of  $G^{**}$  with both endpoints in  $V \setminus X$ . Thus, there are at least  $d$  edges of  $G^{**}$  between  $X$  and  $V \setminus X$ . Hence we must have  $\sum_{v \in X} \beta_v - 2k' \geq d$ .

We have proved that there exist  $b'$  and  $k'$  satisfying the constraints of the lemma, such that  $C(H, G) = \frac{1}{h}(|E| + \frac{1}{2}(b'd + \sum_{v \in X} \beta_v))$ . Hence, by trying all the possible combinations of  $b'$  and  $k'$  satisfying the constraints, we have that  $C(H, G)$  is at least the minimum possible value of (6) subject to the constraints. Computing this minimum can be done in  $O(n)$  time since  $0 \leq k' \leq k(H, G) = O(n)$  and for every possible value of  $k'$  in this range, the minimum possible value of  $b'$  satisfying the constraints can be found in constant time.  $\square$

We will now show that the lower bound for  $C(H, G)$  computed in Lemma 4.7, is, in fact, the exact value of  $C(H, G)$ .

**Lemma 4.8**  $C(H, G)$  is equal to the minimum value of (6) subject to the constraints of Lemma 4.7.

**Proof:** Let  $b'$  and  $k'$  satisfy the constraints of Lemma 4.7, such that (6) is minimal. According to Lemma 4.6, and since  $0 \leq k' \leq k(H, G)$  we know there exists a multigraph  $S$  on the vertices of  $X$ , which satisfies the following three requirements:

1. Each edge of  $S$  is a copy of an edge of  $G$ .
2.  $S$  has  $k'$  edges.
3. Each  $v \in X$  has degree at most  $\beta_v$  in  $S$ .

For  $v \in X$ , define  $\gamma_v = \beta_v - d_S(v)$ , and define  $s = \sum_{v \in X} \gamma_v = \sum_{v \in X} \beta_v - 2k'$ . Our next goal is to create a multigraph  $G^*$  on the vertices of  $V$  which has the property that each edge of  $G^*$  is a copy

of an edge of  $G$ , Each  $v \in X$  has degree  $\beta_v$  in  $G^*$ , and exactly  $b'$  vertices of  $V \setminus X$  have degree  $d$ , while the other  $n - |X| - b'$  vertices are isolated. The multigraph  $S$  will be the sub-multigraph of  $G^*$  induced by  $X$ . It remains to define the other edges of  $G^*$ . Let  $Z = \{z_1, \dots, z_{b'}\}$  be a set of new vertices. (If  $b' = 0$ , then  $Z = \emptyset$ , but in this case also  $\gamma_v = 0$  for all  $v \in X$  by the constraints in Lemma 4.7.)

Let  $\{v_1, \dots, v_{|X|}\}$  be an ordering of  $X$ . For  $i = 1, \dots, |X|$  we perform the following process which assigns edges between  $Z$  and  $X$ : The process assigns  $\gamma_{v_i}$  edges between  $v_i$  and  $Z$  in such a way that after the assignment, the degrees of each pair of vertices of  $Z$  differ by at most 1. Note that the process may introduce multiple edges if some  $\gamma_{v_i}$  is greater than  $b'$ . After the process ends, consider the multigraph  $T$  on the vertices  $X \cup Z$  obtained by the union of  $S$  and the edges assigned between  $X$  and  $Z$ .  $T$  clearly satisfies the following properties:

1. Each  $v \in X$  has degree  $\beta_v$  in  $T$ .
2. If  $b'$  divides  $s$  then each  $z_i$  has degree  $s/b'$  in  $T$ . Otherwise, Exactly  $s \bmod b'$  vertices of  $Z$  have degree  $\lceil s/b' \rceil$  and the other  $b' - (s \bmod b')$  vertices of  $Z$  have degree  $\lfloor s/b' \rfloor$ .

Recall the second constraint in Lemma 4.7, which states that  $s \leq b'd$ . This implies that no vertex of  $Z$  has degree greater than  $d$  in  $T$ .

Our next goal is to add to  $T$  edges between vertices of  $Z$  so that after this addition, the degree of each vertex of  $Z$  will be exactly  $d$ . The sum of the degrees of the vertices of  $Z$  should therefore be  $b'd$  after the addition, while the sum of the degrees of the vertices of  $Z$  in  $T$  is  $s$  prior to the addition. Thus, it suffices to show that there exists a multigraph  $Q$  on  $b'$  vertices, with  $(b'd - s)/2$  edges, such that the degrees of each two vertices of  $Q$  differ by at most one. In fact, in order to show that  $Q$  exists we only need to show that  $(b'd - s)/2$  is a nonnegative integer and that, if  $b'd > s$  then  $b' > 1$  (since for any  $b' \geq 2$  there trivially exists a multigraph on  $b'$  vertices containing as many edges as we want, and with the property that the degrees of any two vertices differ by at most one). The fact that  $b'd \geq s$  follows from the second constraint of Lemma 4.7. The fact that  $(b'd - s)/2$  is an integer follows from the fact that  $bd + \sum_{v \in V} \beta_v$  is even, from the fact that  $b' \equiv b \pmod{2h/d}$ , and from the definition of  $s$  which implies that  $s \equiv \sum_{v \in V} \beta_v \pmod{2}$ . Now if  $b' = 1$  it follows from the third constraint in Lemma 4.7 that  $s \geq d$ , so we must have that if  $b'd > s$  then  $b' > 1$ .

After adding to  $T$  the required set of edges as described in the previous paragraph, we obtain a multigraph  $T'$  on the vertices  $X \cup Z$ , such that  $d_{T'}(v) = \beta_v$  for  $v \in X$  and  $d_{T'}(z) = d$  for  $z \in Z$ . Our goal is to map the vertices of  $Z$  to vertices of  $V \setminus X$ , such that if  $z_i \in Z$  is mapped to some  $u_i \in V \setminus X$  then for each  $(z_i, v) \in T'$  where  $v \in X$ , then  $(u_i, v) \in G$ , and that if  $(z_i, z_j) \in T'$  where  $j < i$ ,

then  $(u_i, u_j) \in G$ . Such a mapping clearly constitutes the required multigraph  $G^*$  (the vertices of  $V \setminus (X \cup \{u_1, \dots, u_{b'}\})$  are the isolated vertices of  $G^*$ ). The mapping can clearly be done in the same manner shown in Lemma 3.9. (One should notice that the inequality  $n - d\epsilon(H)n - |X| - i > 0$  used in Lemma 3.9 is valid here since  $i \leq b'$ , and the minimum value of  $b'$  satisfying the constraints of Lemma 4.7 is clearly at most  $2h/d + \sum_{v \in X} \beta_v < 2h/d + d|X| \ll n$ ).

Having created  $G^*$ , we now apply Lemma 4.1 to  $G^*$ . The conditions of Lemma 4.1 are satisfied since  $\delta(G^*) \leq d < h$ . Therefore, let  $R$  be the multigraph guaranteed by Lemma 4.1. Let  $G'$  be the subgraph of  $G$  obtained by deleting from  $G$  the edges of  $R$  which are not in  $G^*$ . By the same arguments given in Lemma 4.3, we conclude that  $G'$  satisfies the conditions of Lemma 2.1. (The only difference is that we now use  $b'$  instead of  $b$  but since  $b' \equiv b \pmod{2h/d}$ , all the arguments given in Lemma 4.3 still hold). Thus, the union of the  $H$ -decomposition of  $G'$  and the  $H$ -decomposition of  $R$  is an  $H$ -covering of  $G$ . The number of elements in this covering is

$$\frac{e(R) + e(G')}{h} = \frac{|E| + e(G^*)}{h} = \frac{1}{h}(|E| + \frac{1}{2}(b'd + \sum_{v \in X} \beta_v)).$$

□

By lemma 4.8, in order to compute  $C(H, G)$  we need only do the following: First, we compute all the values of  $\beta_v$ . This is done in linear time in the size of  $G$ , namely in  $O(n^2)$  time. We then compute  $k(H, G)$  as shown in Lemma 4.6. This is done in  $O(n^{2.5})$  time. We now compute the minimum of (6) subject to the constraints of Lemma 4.7. This is done in  $O(n)$  time as shown there. This minimum is  $C(H, G)$ , as proved in Lemma 4.8. The overall running time is, therefore,  $O(n^{2.5})$ .

In case the graph  $G$  is  $d$ -divisible, Lemma 4.7 establishes a closed formula for  $C(H, G)$ . Since in this case,  $X = \emptyset$ , and  $k(H, G) = 0$ , Lemma 3.8 states that  $C(H, G)$  is at least the minimum possible value of  $|E|/h + b'd/(2h)$  subject to  $b' \equiv b \pmod{2h/d}$  and  $b' \neq 1$ . Lemma 4.8 shows that  $C(H, G)$  is, in fact, *exactly* the minimum possible value of  $|E|/h + b'd/(2h)$  subject to the above conditions on  $b'$ . Solving this, we get that  $b' = b$  if  $b \neq 1$  and  $b' = b + 2h/d$  if  $b = 1$ . In the first case

$$C(H, G) = |E|/h + bd/(2h) = \lceil |E|/h \rceil = \lceil \frac{\sum_{v \in V} \alpha_v}{2h} \rceil.$$

In the second case,

$$C(H, G) = |E|/h + (b + 2h/d)d/(2h) = |E|/h + bd/(2h) + 1 = \lceil |E|/h \rceil + 1 = \lceil \frac{\sum_{v \in V} \alpha_v}{2h} \rceil + 1.$$

Note that, by the definition of  $b$ ,  $b = 1$  occurs if and only if  $2|E|/d \equiv -1 \pmod{2h/d}$ , which is equivalent to saying that  $h$  divides  $|E| + d/2$ . □

## 5 Concluding remarks, extensions and related problems

We begin this section with several remarks about Theorems 1.1 and 1.2.

1. Theorems 1.1 and 1.2 give a closed formula for computing the  $H$ -packing and  $H$ -covering numbers of dense graphs  $G = (V, E)$  which are either  $d$ -divisible or nowhere  $d$ -divisible, for every fixed graph  $H$ . Fixing  $H$ , in order to compute the formula, we only need to know the degrees of  $G$ , and we therefore have a polynomial algorithm requiring only  $O(V^2)$  time, which computes the  $H$ -packing and  $H$ -covering numbers of  $G$ . In case  $G$  is neither  $d$ -divisible nor nowhere  $d$ -divisible, we can still compute the  $H$ -packing and  $H$ -covering numbers in polynomial time. This fact should be compared with the result of Dor and Tarsi [7], mentioned in the introduction, which implies in particular that for every fixed connected graph  $H$  with at least three edges, it is NP-Hard to compute the  $H$ -Packing and  $H$ -Covering numbers of a general input graph. Thus, there must be restrictions placed on the input graph  $G$ , in order to obtain a closed formula, or a polynomial time algorithm.
2. If  $G$  only satisfies the density constraints, but is neither  $d$ -divisible nor nowhere  $d$ -divisible, we can show that there may be large deviations from the closed formulas in Theorems 1.1 and 1.2. Let  $k$  be any positive integer, and consider, e.g. the packing formula of Theorem 1.1. We shall construct an example of a dense graph  $G$  showing that, e.g.  $P(K_3, G)$  differs from the formula by at least  $k$ . Let  $s \geq 6k$  be an even number. Let  $n$  be any odd integer satisfying  $\epsilon(K_3)n \geq s$ . Now let  $G$  be the  $n$ -vertex complete graph from which the edges of a complete  $s$ -vertex graph have been deleted. Note that  $\delta(G) = n - s \geq (1 - \epsilon(K_3))n$ .  $s$  vertices of  $G$  have degree  $n - s$  which is odd, while  $n - s$  vertices have degree  $n - 1$  which is even. Since  $d = \gcd(K_3) = 2$ ,  $G$  is neither  $d$ -divisible nor nowhere  $d$ -divisible. If we apply the formula of Theorem 1.1 to  $G$  we obtain the value  $(n^2 - n - s^2)/6$ . However, in any packing of  $G$ , every vertex with degree  $n - s$  is incident with at least one uncovered edge, and no such edge is counted twice, since any two vertices with degree  $n - s$  are not adjacent in  $G$ . Thus,

$$P(K_3, G) \leq \frac{e(G) - s}{3} = \frac{n^2 - n - s^2}{6} - \frac{s}{6} \geq \frac{n^2 - n - s^2}{6} - k.$$

Note that in our example  $k$  can even be as large as  $\epsilon(K_3)n/6$ , i.e. a linear function of  $n$ . Similar examples for any other graph  $H$  with  $\gcd(H) \geq 2$  can also be easily constructed. Analogous examples achieving values which are arbitrary larger than the covering formula of Theorem 1.2 can also be constructed. Finally, recall that if  $\gcd(H) = 1$ , then every graph is  $d$ -divisible, and therefore Theorems 1.1 and 1.2 do not impose any divisibility restrictions on  $G$ .

3. The algorithm we have presented for computing  $P(H, G)$  for general dense graphs, runs in polynomial time, but the degree of the polynomial is not small (it is at least 5). It would be interesting to decide whether there exists an algorithm which is considerably faster. Such an algorithm would probably need to avoid the application of the maximum weighted matching algorithm that we have used.

We now turn our attention to related problems and extensions of Theorems 1.1 and 1.2.

1. Let  $L$  be a maximum packing of  $H$  in  $G$ . The *leave-graph* of  $L$  is the graph induced by all the edges of  $G$  that are not covered by a member of  $L$ . The structure of the leave-graph of  $H$  in  $G = K_n$  has been studied in [6, 11] and all known results are for special cases of  $H$ . The proof of Theorem 1.1 constructs a maximum packing, where the leave graph, denoted by  $G^*$  in the proof, has, in certain cases, an imposed degree constraint. Namely, in case  $G$  is nowhere  $d$ -divisible, each vertex  $v \in V$  must have degree  $\nu_v$  in the leave-graph (recall the definition of  $\nu_v$  in the proof of Lemma 3.3). In case  $G$  is  $d$  divisible, we also know the exact degree sequence of the leave-graph, i.e  $b$  vertices with degree  $d$  and  $n - b$  vertices with degree 0 (except for the special case where  $1 \leq b \leq d$  in which case there are  $b + 2h/d$  vertices with degree  $d$  and  $n - b - 2h/d$  vertices with degree 0). Since these are the only restrictions placed on  $G^*$  in the proof, any spanning subgraph of  $G$  realizing these degree requirements forms a leave-graph. Consequently, Theorem 1.1 completely solves the leave-graph problem for all graphs  $G$  which are either  $d$ -divisible or nowhere  $d$ -divisible. In particular, the structure of the leave-graphs for  $K_n$  is determined, provided  $n$  is large enough.
2. Let  $L$  be a minimum covering of  $H$  in  $G$ . The *excess-graph* of  $L$  is the *multigraph* induced by the edges of  $G$  which appear in more than one copy of  $L$ . The multiplicity of an edge in the excess-graph is one less than the number of copies of  $L$  in which it appears. The structure of the excess-graph of  $H$  in  $G = K_n$  has been studied in [6, 12] and all known results are for special cases of  $H$ . As in the case of the leave-graph, the proof of Theorem 1.2 constructs a minimum covering, where the excess-graph, denoted by the multigraph  $G^*$  in the proof, has an imposed degree constraint, in case  $G$  is either  $d$ -divisible or nowhere  $d$ -divisible. These degree constraints are completely determined in the theorem, and any multigraph which meets these degree constraints is a valid excess-graph. Consequently, Theorem 1.2 completely solves the excess-graph problem for all graphs  $G$  satisfying the conditions of the theorem. In particular, the structure of the excess-graphs for  $K_n$  is determined, provided  $n$  is large enough.
3. The *overlap* of an  $H$ -covering  $L$  of  $G$  is defined as the maximum number of appearances of an edge in members  $L$ . It is known [2] that if  $n \geq n(H)$ , then there exists an  $H$ -covering of

$K_n$  with overlap at most 2. Etzion [2] has conjectured that  $CO(H, K_n) - C(H, K_n) \leq c(H)$  where  $CO(H, G)$  is the minimum number of copies in an  $H$ -covering of  $G$  with overlap 2, and  $c(H)$  is a constant depending only on  $H$ . This conjecture has been solved in [4]. Theorem 1.2 shows that Etzion's conjecture is also valid for many graphs which are not complete. To see this, note that, by Corollary 4.4,  $CO(H, G) = C(H, G)$  in case at least  $n/(10d^3)$  vertices are not divisible by  $d$ , which includes the case where  $G$  is nowhere  $d$ -divisible. Also, the multigraph  $T'$  constructed in Lemma 4.8 can, in fact, be a graph, if, say,  $X = \emptyset$  and if we allow  $b'$  to be larger than  $d$  (as in the proof of the lemma 3.9). This can be done by adding  $2h/d$  to  $b'$  in case  $1 \leq b' \leq d$ . This addition still maintains the constraints of Lemma 4.7. The effect of adding  $2h/d$  to  $b'$  is an increase of  $h$  to the number of edges of  $G^*$  and, thus, the covering obtained is one greater than the optimal, but has the advantage that every edge is covered at most twice. It follows that  $CO(H, G) \leq C(H, G) + 1$  in case  $G$  is  $d$ -divisible. This clearly solves and sharpens the problem posed by Etzion, and extends it to a large class of graphs  $G$ .

## 6 Acknowledgment

The authors wish to thank Alex Rosa for his encouragement.

## References

- [1] B. Bollobás, *Extremal Graph Theory*, Academic Press, 1978.
- [2] Y. Caro, Y. Roditty and Y. Schonheim, *Covering the edges of the complete graph with minimum overlap*, submitted.
- [3] Y. Caro and R. Yuster, *Packing graphs: The packing problem solved*, Elect. J. of Combin. 4 (1997), #R1.
- [4] Y. Caro and R. Yuster, *Covering graphs: The covering problem solved*, submitted.
- [5] Y. Caro and R. Yuster, *Efficient covering designs of the complete graph*, Elect. J. of Combin. 4 (1997), #R10.
- [6] C.J. Colbourn and J.H. Dinitz, *CRC Handbook of Combinatorial Design*, CRC press 1996.
- [7] D. Dor and M. Tarsi, *Graph decomposition is NPC - A complete proof of Holyer's conjecture*, Proc. 20th ACM STOC, ACM Press (1992), 252-263.

- [8] S. Even and O. Kariv, *An  $O(n^{2.5})$  algorithm for maximum matching in general graphs*, Proc. 16th IEEE FOCS, (1975), 100-112.
- [9] Z. Galil, S. Micali and H.N. Gabow, *An  $O(EV \log V)$  algorithm for finding a maximal weighted matching in general graphs*, Siam J. Comput. 15 (1986), 120-130.
- [10] T. Gustavsson, *Decompositions of large graphs and digraphs with high minimum degree*, Doctoral Dissertation, Dept. of Mathematics, Univ. of Stockholm, 1991.
- [11] D.G. Hoffman, C.C. Lindner, M.J. Sharry and A.P. Street, *Maximum packing of  $K_n$  with copies of  $K_4 \setminus \{e\}$* , Aeq. Math. 51 (1996), 247-269.
- [12] C.C Lindner and A.P. Street, *Simple maximum coverings of  $K_n$  with copies of  $K_4 \setminus \{e\}$* , Aeq. Math. 52 (1996), 284-301.
- [13] W.T. Tutte, *The factors of graphs*, Canad. J. Math. 4 (1952), 314-328.