NON-REPETITIVE COLORINGS OF GRAPHS

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Abstract. A sequence $a = a_1a_2...a_n$ is said to be *non-repetitive* if no two adjacent blocks of a are exactly the same. For instance the sequence 1**2323**21 contains a repetition 2323, while 123132123213 is non-repetitive. A theorem of Thue asserts that, using only three symbols, one can produce arbitrarily long non-repetitive sequences. In this paper we consider a natural generalization of Thue's sequences for colorings of graphs. A coloring of the set of edges of a given graph G is *non-repetitive* if the sequence of colors on any path in G is non-repetitive. We call the minimal number of colors needed for such a coloring the *Thue number* of G and denote it by $\pi(G)$.

The main problem we consider is the relation between the numbers $\pi(G)$ and $\Delta(G)$. We show, by an application of the Lovász Local Lemma, that the Thue number stays bounded for graphs with bounded maximum degree, in particular, $\pi(G) \leq c\Delta(G)^2$ for some absolute constant c. For certain special classes of graphs we obtain linear upper bounds on $\pi(G)$, by giving explicit colorings. For instance, the Thue number of the complete graph K_n is at most $2n-3$, and $\pi(T) \leq 4(\Delta(T)-1)$ for any tree T with at least two edges. We conclude by discussing some generalizations and proposing several problems and conjectures.

1. Introduction

The problem we consider in this paper emerged as a graph theoretical variant of the non-repetitive sequences of Thue. A finite sequence $a = a_1a_2...a_n$ of symbols from a set S is called *non-repetitive* if it does not contain a sequence of the form $xx = x_1x_2...x_mx_1x_2...x_m$, $x_i \in S$, as a subsequence of *consecutive* terms. For instance the sequence $a = 123132123213$ over the set $S = \{1,2,3\}$ is non-repetitive, while $b = 1\mathbf{2323}21$ is not. A beautiful theorem of Thue [23] asserts that there exist arbitrarily long non-repetitive sequences built of only three different symbols. The method discovered by Thue is constructive and uses substitutions over a given set of symbols. For instance, the substitution

$$\begin{aligned} 1 &\rightarrow 12312 \\ 2 &\rightarrow 131232 \\ 3 &\rightarrow 1323132 \end{aligned}$$

preserves the property of non-repetitiveness on the set of finite sequences over $\{1,2,3\}$. That is, replacing all symbols in a non-repetitive sequence by the assigned blocks results in a sequence that still does not contain repetitions. In consequence, starting from any single symbol one can produce, by iteration of this procedure, a

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FIGURE 1. The Thue number of the Petersen graph is 5.

non-repetitive sequence of any given length. Note that this fact implies also the existence of an *infinite* non-repetitive sequence over a 3-element set of symbols.

Sequences generated by substitutions have found many unexpected applications in such diverse areas as group theory, universal algebra, number theory, dynamical systems, ergodic theory, formal language theory, etc. (see [2], [5]-[7], [12], [16]-[19], [21], [22]). Perhaps the most spectacular was their use in the solution of the famous Burnside problem for groups (see [2], [17]). Also, many generalizations of such sequences have been considered. For example, a long standing open problem of Erdös [10] and Brown [4] concerns the existence of an infinite *strongly non-repetitive* sequence on four symbols. In such a sequence no two adjacent blocks are *permutations* of each other. Note that, as in the original case of Thue sequences, it is not *a priori* clear that any finite number of symbols will do. However, Evdokimov [11] proved that the goal can be achieved on 25 symbols and Pleasants [20] and Dekking [8] lowered this number to 5. Finally, with the help of computer, Keränen [15] found a substitution with blocks of length 85 preserving strong non-repetitiveness on 4 symbols, which is best possible.

Another direction of generalizations, undertaken by Bean, Ehrenfeucht and McNulty [2], has a continuous flavor. A coloring $f : \mathbb{R} \to S$ of the real line is said to be *square-free* if no two adjacent intervals are colored along the same pattern. In other words, for any two points $x < y$ there is a point $z \in (x, y)$, such that $f(z) \neq f(z + y - x)$. It was proved in [2] that there exist 2-colorings of $\mathbb{R}$ satisfying the above property. More striking results of this type can be found in [13] and [14]. For instance, by transfinite induction one can prove that there exists a 2-coloring of the plane such that no two different topological disks D and D' are colored *similarly*, i.e. for any homeomorphism h transforming D onto D' there is a point $x \in D$ colored differently from its image $h(x)$.

In the present paper we propose another variation on the non-repetitive theme, this time concerning graphs. A coloring of the set of edges of a graph G is called *non-repetitive* if the sequence of colors on any path in G is non-repetitive. We call the minimal number of colors needed the *Thue number* of G and denote it by $\pi(G)$. As an example consider the Petersen graph P. In Fig.1 we show a non-repetitive 5-coloring of the edges of P. Note that repetitions forming full cycles are allowed. Since, as can easily be checked, 4 colors do not suffice for this task, we have $\pi(P) = 5$.

The main question we investigate here is how the Thue number $\pi(G)$ depends on the maximum degree $\Delta = \Delta(G)$ of a graph G. The theorem of Thue says simply that $\pi(P_n) = 3$, for all $n \geq 4$, where P_n denotes a path with n edges. An immediate consequence is that $\pi(C_n) \leq 4$, for all $n \geq 3$—simply color a spanning path in C_n by the first three colors and the last edge by the fourth color. Although it is natural to suspect that this coloring is not optimal, there are cycles for which three colors do not suffice. For instance, one can easily check that $\pi(C_5) = 4$. Of course, the result for cycles gives that $\pi(G) \leq 4$ for all graphs with $\Delta \leq 2$, and it is natural to wonder whether $\pi(G)$ is bounded on the class of graphs with $\Delta \leq k$, for each $k \geq 3$. In Section 2 we will prove, by the probabilistic method, that this is indeed the case.

More specifically, by an application of the Lovász Local Lemma, we obtain that

$$\pi(G) \leq c\Delta^2,$$

for all graphs G with maximum degree at most Δ, where c is an absolute constant. In Section 3 we will use explicit colorings to get better estimates for some special classes of graphs. For instance, the Thue number of the n-dimensional grid is at most $3n$, and for the complete graph K_n the Thue number is at most $2n-3$.

The final section of the paper is devoted to open problems and conjectures as well as the discussion of some possible generalizations.

2. Bounding $\pi(G)$; an application of the Lovász Local Lemma

In this section we prove the upper bound for $\pi(G)$ mentioned in the introduction. It will be convenient to use the following version of the Lovász Local Lemma, which is in fact equivalent to the standard asymmetric version (see [1]).

Lemma 1. *(The Local Lemma; Multiple Version) Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup ... \cup \mathcal{A}_r$ be a partition of a finite set of events $\mathcal{A}$, with $\Pr(A) = p_i$ for every $A \in \mathcal{A}_i$, $i = 1, 2, ..., r$. Suppose that there are real numbers $0 \leq a_1, a_2, ...a_r < 1$ and $\Delta_{ij} \geq 0$, $i, j = 1, 2, ..., r$ such that the following conditions hold:*

(i) for any event $A \in \mathcal{A}_i$ there exists a set $\mathcal{D}_A \subseteq \mathcal{A}$ with $|\mathcal{D}_A \cap \mathcal{A}_j| \leq \Delta_{ij}$ for all $j = 1, 2, ..., r$, such that A is independent of $\mathcal{A} \setminus (\mathcal{D}_A \cup \{A\})$,

(ii) $p_i \leq a_i \prod_{j=1}^{r} (1 - a_j)^{\Delta_{ij}}$ for all $i = 1, 2, ..., r$.

Then $\Pr(\bigcap_{A \in \mathcal{A}} \bar{A}) > 0$.

We apply this Lemma to prove that Thue number is bounded on the class of graphs with bounded maximum degree.

Theorem 1. *There exists an absolute constant c such that*

$$\pi(G) \leq c\Delta^2,$$

for all graphs G with maximum degree at most Δ.

Proof. Let G be a simple graph with maximum degree Δ. Randomly color the edges of G with C colors, where C is a number to be specified later. For a path P in G of even length let $A(P)$ denote the event that the second half of P is colored the same as the first. In other words, the sequence of colors on P forms a repetition. Set $\mathcal{A}_i = \{A(P) : P \text{ is a path of length } 2i \text{ in } G\}$, so $p_i = C^{-i}$. Since each path of length $2i$ shares an edge with not more than $4ij\Delta^{2j}$ paths of length $2j$, we may take $\Delta_{ij} = 4ij\Delta^{2j}$.

Let $a_i = a^{-i}$, with $a = 2\Delta^2$. Then $(1 - a_i) \geq e^{-2a_i}$, as $a_i \leq 1/2$, and the Lemma applies provided

$$p_i \leq a_i \prod_j e^{-2a_j \Delta_{ij}},$$

that is,

$$C^{-i} \leq a^{-i} \prod_j e^{-8ija^{-j}\Delta^{2j}},$$

or

$$C \geq a \exp\left(8 \sum_j \left(\frac{\Delta^2}{a}\right)^j j\right).$$

Substituting $a = 2\Delta^2$ into the last inequality, as the series $\sum_{j=1}^{\infty} \frac{j}{2^j}$ converges to 2 we see that if $C \geq 2e^{16}\Delta^2$ the Local Lemma guarantees the existence of a non-repetitive C-coloring. This proves Theorem 1 with $c = 2e^{16} + 1$. ∎

3. Explicit Colorings

In this section we consider special classes of graphs for which we can give good bounds on the Thue number by giving explicit non-repetitive colorings. The first example is the class of complete graphs.

Proposition 1. *$\pi(K_{2^k}) = 2^k - 1$ for all $k \geq 1$. Therefore, $\pi(K_n) \leq 2n - 3$.*

Proof. First, label the vertices of K_{2^k} by distinct elements of the additive group $\mathbb{Z}_2^k$, the direct product of k copies $\mathbb{Z}_2$. Next, color the edges by non-zero elements of $\mathbb{Z}_2^k$, so that an edge with vertices labeled by x and y gets color $x + y$. It is easily seen that repetitions appear only on cycles, since $x + x = 0$. ∎

For the second example we need a non-repetitive sequence which is also *palindrome-free.* A sequence $a = a_1a_2...a_n$ is a *palindrome* if it is equal to its own reflection $\tilde{a} = a_na_{n-1}...a_1$, as, for instance, 1231321. It is easy to see that on three symbols repetitions and palindromes cannot be avoided simultaneously in long sequences. On four symbols they can; simply take an arbitrary non-repetitive sequence on the set $\{1, 2, 3\}$ and insert the fourth symbol between consecutive blocks of length 2. For example, from the sequence 123132123213 one gets 124314324124324134.

Proposition 2. *Let T be any tree with $\Delta(T) \geq 2$. Then $\pi(T) \leq 4(\Delta(T) - 1)$.*

Proof. Let T be a tree of maximum degree $\Delta \geq 2$. Choose a vertex of degree strictly less than Δ as the root of T and arrange the rest of vertices by their distance from the root. The edges of T can be naturally partitioned into levels, $L_1, L_2, \ldots$, each of which consists of disjoint stars. Let $b = b_1b_2\ldots$ be a non-repetitive and palindrome-free sequence over the set $\{1, 2, 3, 4\}$, which we will assign to the levels of T. Take four disjoint sets of colors $A_i = \{i', i'', ..., i^{(\Delta-1)}\}$, $i = 1, 2, 3, 4$, and colour each star on level j by distinct colors from the set A_{b_j}. For an example of such coloring see Fig.2.

It is easy to see that this coloring is non-repetitive: suppose that $a = a_1a_2...a_{2n}$ is a sequence of colors on some path $P \subseteq T$ forming a repetition. Note that $n \geq 2$. Also, since b is non-repetitive, P cannot be monotone, i.e., it must have two consecutive edges belonging to the same star, with colors from the same set A_i. Since this situation may happen only once in P, it must take place in the middle

FIGURE 2. $\pi(T) \leq 4(\Delta(T) - 1)$.

of P. Consider the sequence $c = c_1 c_2 ... c_{2n}$ defined by $c_k = j$ if $a_k \in A_j$. Since the first half of a is the same as the second half, the same is true of c. Also, from the shape of P, the second half of c is the reverse of the first. Thus $c_1 c_2 \dots c_n$ is a palindrome, contradicting the assumption that b is palindrome-free, and completing the proof. ∎

In the next proposition we estimate the Thue number of the n-fold Cartesian product of trees. The *Cartesian product* $G \times G'$ is the graph with vertex set $V(G) \times V(G')$ in which the vertex (v, v') is adjacent to the vertex (w, w') whenever $v = w$ and v' is adjacent to w', or $v' = w'$ and v is adjacent to w. Since this operation is associative, $G_1 \times ... \times G_n$ is well defined for $n \geq 2$. When all of the G_i are paths we obtain the n-dimensional grid.

First, we prove a simple general observation concerning arbitrary walks in non-repetitively colored graphs.

Lemma 2. *Let $f : E(G) \to \{1, 2, ..., k\}$ be a non-repetitive coloring of the set of edges of a graph G. Suppose $W = e_1 e_2 ... e_{2m}$, $m \geq 1$, is an acyclic walk in G such that $f(e_i) = f(e_{i+m})$, for all $i = 1, 2, ..., m$. Then W must be closed.*

Proof. We apply induction on m. For $m = 1$ the assertion is clear. Assume it holds for all walks of length at most $2m - 2$. Consider a pair $e_j e_{j+1}$ of adjacent edges of W such that $e_j = e_{j+1}$. Such a pair must exist, since otherwise W is a simple path, which contradicts the assumptions. If $1 \leq j < m$ then also $e_{j+m} = e_{j+1+m}$, by assumption. So, by removing the edges e_j, e_{j+1}, e_{j+m}, e_{j+1+m} from the walk W we get a shorter walk W' with repetitive color pattern. Thus W', and consequently W, is closed. Similarly for $m < j < 2m$.

In the rest of the proof we do not use induction. It remains to consider the situation in which $j = m$ and there are no other places in W where two adjacent edges are equal. Then we can write $W = xy\tilde{y}z$, where $\tilde{y}$ is the reflection of y, and $P = xz$ is a simple path. Denote by l the length of the sequence y, and by $a = f(y)$ the sequence of colors on the edges of y. We will show that the sequence of colors $f(P)$ on the path P always contains a repetition in the middle, the length of which depends on l, unless P is empty. In fact, if $l < m/2$ we may write

$$f(W) = f(x)f(y)f(\tilde{y})f(z) = \tilde{a}ba\tilde{a}ba,$$

for some non-empty sequence b. Hence, in this case we get

$$f(P) = \tilde{a}bba.$$

Otherwise, if $m/2 \leq l < m$ we have $f(x) = f(\tilde{z})$ which means that the colors of two edges in the middle of P are identical, completing the proof. ∎

Proposition 3. *Let $T_1, T_2, ..., T_n$ be trees. Then $\pi(T_1 \times T_2 \times ... \times T_n) \leq \sum_{i=1}^{n} \pi(T_i)$. In particular, if G is an n-dimensional grid then $\pi(G) \leq 3n$.*

FIGURE 3. Thue number of a plane grid is at most 6.

Proof. Let $A_1, A_2, ..., A_n$ be pairwise disjoint sets of colors such that $|A_i| = \pi(T_i)$ for $i = 1, 2, ..., n$. The graph $T = T_1 \times T_2 \times ... \times T_n$ may be decomposed into "parallel" copies of the graphs T_i in a standard manner. Color each copy of T_i non-repetitively by colors from A_i, so that the corresponding edges get the same color (see Fig.3). Now, if P is a path in T then its "projection" on a fixed copy of T_i is a walk and the assertion follows from Lemma 2. ∎

4. Final Discussion

First, let us point out that the idea of applying probabilistic methods in Thue-like problems is not new. There are two earlier results establishing rather strong avoidability properties for infinite binary sequences using probabilistic arguments. One of them is the theorem of Beck [2] asserting, for any $\varepsilon > 0$, the existence of an infinite binary sequence in which two identical blocks of length $n > n_0(\varepsilon)$ are separated by at least $(2-\varepsilon)^n$ terms. The other is an exercise in the book of Alon and Spencer [1], and says that for any $\varepsilon > 0$ there is an infinite binary sequence in which two adjacent blocks of length $m > m_0(\varepsilon)$ differ in at least $(\frac{1}{2} - \varepsilon)m$ places. Both results rely on the Lovász Local Lemma and both imply the existence of infinite non-repetitive sequences over some finite set of symbols, although the resulting number of symbols is much bigger than 3.

4.1. **Asymptotics of $\pi(n)$.** Actually, the main subject of our interest is the asymptotic shape of the function $\pi(n)$ defined by

$$\pi(n) = \max\{\pi(G);\ \Delta(G) \leq n\}.$$

It should be stressed that for $n \geq 3$ the situation is very different from the original case of sequences. The quadratic upper bound on $\pi(n)$, obtained probabilistically in Theorem 1, is the best known at this moment. Actually, we do not know of any other argument proving finiteness of $\pi(n)$, for any $n \geq 3$. On the other hand, the results of Section 3 suggest that perhaps some general constructive method is possible, giving better estimates. Anyway, we propose the following conjecture.

Conjecture 1. *There is an absolute constant c such that*

$$\pi(n) \leq cn,$$

for all $n \geq 1$.

Obviously, non-trivial lower bounds for $\pi(n)$ are also desired. Here, one may look for bad graphs among members of some special families. Good candidates seem to be complete graphs with $2^k + 1$ vertices. For instance, one can check that $\pi(K_5) = 7$, which is quite a jump from $\pi(K_4) = 3$.

4.2. **Concrete problems.** One concrete problem in this area has been touched on in the introduction. It concerns the Thue number of cycles. We know that $\pi(C_n) \leq 4$, and it is easy to see that $\pi(C_5) = 4$. By numerical experiment we have found that the same happens also for $n = 7, 9, 10, 14$ and 17, and for no other value up to 2001. This justifies the following conjecture.

Conjecture 2. *$\pi(C_n) = 3$, for all $n \geq 18$.*

It was proved in [2] by the method of substitutions that there are infinitely many n for which $\pi(C_n) = 3$. Our experiment suggests also that the number of non-equivalent non-repetitive colorings of C_n grows exponentially with n. For sequences it is well known that this is the case (see [5]). The best known lower bound on the number of ternary non-repetitive sequences of length n is $2^{n/17}$, due to Ekhad and Zeilberger [9].

It would be also nice to know the exact value of Thue number for other graphs. By Proposition 2 we have $\pi(T) \leq 8$, for any binary tree T, but this seems to be an overestimate. From Proposition 3 it follows that $\pi(\mathbb{Z}^2) = 5$ or 6, but which is the correct number?

4.3. **Generalizations.** There are many exciting generalizations of non-repetitive sequences and for most of them it also make sense to study their graph theoretical variants. In principle, any property of sequences can be translated into a property of graphs, via colored paths. In particular, one may take any *avoidable pattern* (see [2], [5]-[7], [17]) and study its behavior on graphs. For instance, the property of strong non-repetitiveness, mentioned briefly in the introduction, also leads to similar intriguing problems.

In another direction, one may look at different types of colorings. Certainly, it could be equally interesting to consider *vertex non-repetitive colorings.* Obviously, Theorem 1 remains true in this case, but for some classes of graphs the *vertex Thue number* may be bounded, even if Δ is arbitrarily large. For example, for any tree four colors suffice; this can be proved like Proposition 2, by using a palindrome-free non-repetitive sequence on four symbols. Is it true that the vertex Thue number of planar graphs is bounded? We conclude this paper with the following theorem, that shows that for vertex coloring, the quadratic dependence of the number of colors, provided by Theorem 1 (for vertex coloring) is nearly tight.

Theorem 2. *There exists an absolute constant $c > 0$ with the following property. For every integer $\Delta > 1$, there exists a graph G with maximum degree Δ such that every non-repetitive vertex coloring of G uses at least $c\frac{\Delta^2}{\log \Delta}$ colors.*

Proof. The proof is probabilistic, and we make no attempt to optimize the absolute constant c it provides. We also omit all floor and ceiling signs whenever these are not crucial, to simplify the presentation. Clearly, it suffices to prove the assertion for large values of Δ (as the result for small Δ is trivial if c is sufficiently small). Let n be a (large) number, define $p = \frac{10\sqrt{\log n}}{\sqrt{n}}$, and let $G = G(n, p)$ be the random graph on a set V of n labelled vertices obtained by choosing each pair of vertices to be an edge, randomly and independently, with probability p. We first claim that almost surely (that is, with probability that tends to 1 as n tends to infinity), G satisfies the following three properties.
(i) The maximum degree of G, denoted by Δ, satisfies, $\Delta \leq 20\sqrt{n \log n}$.

(ii) There is at least one edge of G between any two disjoint sets of size at least $n/12$ each.
(iii) For every collection of $m = n/4$ pairwise disjoint subsets

$$\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_m, v_m\}$$

of V, there is an $S \subset \{1, 2, \ldots, m\}$, $|S| \geq m/3 = n/12$, such that the graph on the set of vertices $\{s : s \in S\}$ in which st is an edge iff both $u_s u_t$ and $v_s v_t$ are edges of G is connected.

To prove this claim note first that (i) and (ii) are trivial. To prove (iii) fix $m = n/4$ pairwise disjoint sets as above, and let us estimate the probability that there is no S as required. Let H be the graph on the set of vertices $\{1, 2, \ldots, m\}$ in which ij is an edge iff both $u_i u_j$ and $v_i v_j$ are edges of G. Obviously this is a random graph with $m = n/4$ vertices in which every pair of vertices forms an edge, randomly and independently, with probability $p^2 = \frac{100 \log n}{n}$. Our objective is to estimate the probability that there is no connected component of at least $m/3 = n/12$ vertices in H. But if this happens then the set of vertices of H can be partitioned into two disjoint sets, each of size at least $m/3$ and at most $2m/3$, with no edges between them. The probability of this event is at most $2^m (1 - \frac{100 \log n}{n})^{2m^2/9} < n^{-n}$. Since the number of possible choices for the m sets $\{u_i, v_i\}$ is smaller than $n^{n/2}$ we conclude that the probability that the assertion of (iii) fails is at most $n^{-n/2}$ completing the proof of the claim.

Returning to the proof of the theorem let G be a graph satisfying all three properties (i),(ii),(iii) above. To complete the proof we show that in any vertex coloring of G by at most $n/2$ colors, there is a path which is colored repetitively. Given such a coloring f, omit, first, one vertex from each color class containing an odd number of vertices, and partition the remaining vertices into pairs, where each pair of vertices has the same color. Clearly this produces at least $m = n/4$ pairs. Let $\{u_1, v_1\}, \ldots, \{u_m, v_m\}$ be m of these pairs. By property (iii) there is an $S \subset \{1, 2, \ldots, m\}$ satisfying the assertion of (iii), and by (ii) applied to the two sets $\{u_t : t \in S\}$ and $\{v_s : s \in S\}$ there is an edge $u_t v_s$ of G with $s, t \in S$. Let $s = s_1, s_2, \ldots, s_r = t$ be a path from s to t in the graph on S in which ij is an edge iff both $u_i u_j$ and $v_i v_j$ are edges. (Such a path exists, by (iii)). Then, the path $u_s u_{s_2} u_{s_3} \ldots u_t v_s v_{s_2} v_{s_3} \ldots v_t$ shows that the coloring is **not** non-repetitive, and completes the proof of the theorem. ∎

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