A note on the degree, size and chromatic index of a uniform hypergraph

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Abstract

Let $\mathcal{H}$ be a k-uniform hypergraph in which no two edges share more than t common vertices, and let D denote the maximum degree of a vertex of $\mathcal{H}$. We conjecture that for every $\epsilon > 0$, if D is sufficiently large as a function of t, k and ϵ, then the chromatic index of $\mathcal{H}$ is at most $(t - 1 + 1/t + \epsilon)D$. We prove this conjecture for the special case of intersecting hypergraphs in the following stronger form: If $\mathcal{H}$ is an intersecting k-uniform hypergraph in which no two edges share more than t common vertices, and D is the maximum degree of a vertex of $\mathcal{H}$, where D is sufficiently large as a function of k, then $\mathcal{H}$ has at most $(t - 1 + 1/t)D$ edges.

1 Introduction

For a k-uniform hypergraph $\mathcal{H}$ (which may have multiple edges), let $D(\mathcal{H})$ denote the maximum degree of a vertex of $\mathcal{H}$, and let $\chi'(\mathcal{H})$ denote the chromatic index of $\mathcal{H}$, that is, the minimum number of colors needed to color the edges of $\mathcal{H}$ so that each color class forms a matching. For an integer t satisfying $1 \le t \le k$, we say that $\mathcal{H}$ is *t-simple* if every two distinct edges of $\mathcal{H}$ have at most t vertices in common. We propose the following conjecture.

Conjecture 1.1 *For every $k \ge t \ge 1$ and every $\epsilon > 0$ there is a finite $D_0 = D_0(k, t, \epsilon)$ so that if $D > D_0$ then every k-uniform, t-simple hypergraph $\mathcal{H}$ with maximum degree at most D satisfies*

$$\chi'(\mathcal{H}) \le (t - 1 + 1/t + \epsilon)D. \tag{1}$$

For $k = t = 1$ this is trivially true. For $k = 2$ and $t = 1$, Vizing's theorem [7] implies that $\chi'(\mathcal{H}) \le D + 1$, showing the assertion holds in this case as well. For $k = t = 2$, Shannon's theorem [6] implies that $\chi(\mathcal{H}) \le \lfloor 3D/2 \rfloor$ and hence (1) holds in this case too. For $k > t = 1$ the validity of the conjecture follows from the main result of Pippenger and Spencer in [5]. All other cases are open. It

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is worth noting that the special case $k = t$ has been conjectured by Füredi, Kahn and Seymour in [4], where they prove a fractional version of this case (as well as a more general result).

It is not difficult to see that the conjecture, if true, is tight for every $k \geq t$ for which there exists a projective plane of order $t-1$. Here, for $t = 1$ such a plane is, by definition, a single point and a single line containing it, and for $t = 2$ it consists of the three lines and three points of a triangle. To see that the conjecture is tight when the required plane exists, let D be a large integer divisible by t, define $m = t^2 - t + 1$ and fix a projective plane of order $t-1$ with m lines $l_1, l_2, \cdots, l_m$ on a set of m points. For each of the lines l_i, let $\mathcal{F}_i$ be a collection of D/t sets of size k containing l_i, so that all the mD/t sets $\{A - l_i : 1 \leq i \leq m \text{ and } A \in \mathcal{F}_i\}$ are pairwise disjoint. Let $\mathcal{H}$ be the k-uniform hypergraph consisting of all the sets in all the families $\mathcal{F}_i$. Then $\mathcal{H}$ is intersecting, k-uniform and t-simple, its maximum degree is D and it has $mD/t = (t - 1 + 1/t)D$ edges.

The above conjecture seems difficult. In the present note we only make some modest progress in its study, by proving it for the special case of *intersecting* hypergraphs. Note that since the chromatic index of an intersecting hypergraph is simply the number of its edges the conjecture in this case reduces to a statement about the maximum possible number of edges of a t-simple, k-uniform intersecting hypergraph with a given maximum degree. In order to state our main result we need an additional definition. A collection $\mathcal{F}$ of r edges in a k-uniform hypergraph is called a Δ-*system of size* r if all the intersections $A \cap B$ for $A, B \in \mathcal{F}$, $A \neq B$ are the same. In this case, the common value of such an intersection is called the *core* of the system. Note that the core, call it C, is the intersection of all edges of the system, and the sets $\{A - C : A \in \mathcal{F}\}$ are pairwise disjoint.

Erdös and Rado [2] proved that any k-uniform hypergraph with more than $(r-1)^k \cdot k!$ edges contains a Δ-system of size r. Let $f(k)$ denote the maximum number of edges of a k-uniform hypergraph which contains no Δ-system of size $k+1$. By the above result, $f(k) \leq k^k \cdot k!$, and although there are some better bounds known we omit them, as for our purposes here any finite bound suffices.

The following result shows that the assertion of Conjecture 1.1 holds for intersecting hypergraphs.

Theorem 1.2 *Suppose $\mathcal{H}$ is a k-uniform, t-simple intersecting hypergraph with maximum degree $D = D(\mathcal{H}) > tf(k)$. Then the number of edges of $\mathcal{H}$ is at most $(t - 1 + 1/t)D$.*

The proof is short and is presented in the next section. The final section contains some remarks and further problems.

2 The proof

Let $\mathcal{H}$ be a k-uniform, t-simple intersecting hypergraph. We need the following two easy lemmas, which are both valid even without the assumption that $\mathcal{H}$ is t-simple.

Lemma 2.1 *If $\mathcal{F} \subseteq \mathcal{H}$ is a Δ-system of size $\geq k+1$ with core C, then*

$$C \cap A \neq \emptyset, \quad \textit{for all } A \in \mathcal{H}.$$

Proof. Suppose there is an edge $A \in \mathcal{H}$ such that $C \cap A = \emptyset$. Then, since $\mathcal{H}$ is intersecting, A has a non-empty intersection with each $F \setminus C$ for $F \in \mathcal{F}$. But, since the sets $F \setminus C$ are pairwise disjoint, $|A|$ must be at least $|\mathcal{F}| \geq k+1$, which is a contradiction.

□

Lemma 2.2 *Suppose $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{H}$ are Δ-systems of sizes $\geq k+1$ with cores C_1 and C_2, respectively. Then $C_1 \cap C_2 \neq \emptyset$.*

Proof. If $C_1 \cap C_2 = \emptyset$, then Lemma 2.1 implies that $(F \setminus C_1) \cap C_2 \neq \emptyset$ for all $F \in \mathcal{F}_1$. Since the sets $F \setminus C_1$ are pairwise disjoint, $|C_2| \geq |\mathcal{F}_1| \geq k+1$, a contradiction.

□

We also need the following result of Füredi [3], which shows that the assertion of Theorem 1.2 holds for $t = k$ (in a somewhat sharper form).

Theorem 2.3 *If $\mathcal{G}$ is a t-uniform intersecting hypergraph, then*

$$|\mathcal{G}| \leq (t - 1 + 1/t)D(\mathcal{G}) \ .$$

Moreover, if there is no finite projective plane among the subhypergraphs of $\mathcal{G}$, then $|\mathcal{G}| \leq (t-1)D(\mathcal{G})$.

Proof of Theorem 1.2. Take a maximal edge disjoint family $\{\mathcal{F}_1, \cdots, \mathcal{F}_r\}$ of Δ-systems in $\mathcal{H}$ of sizes $\geq k+1$. Let $C_1, \cdots, C_r$ be the corresponding cores. Then each $|C_i| \leq t$ and by the definition of $f(k)$

$$\sum_i |\mathcal{F}_i| \geq |\mathcal{H}| - f(k) \ . \tag{2}$$

If $|C_i| < t$ for some i, then Lemma 2.1 implies that

$$|\mathcal{H}| \leq |C_i| D \leq (t-1)D \ .$$

Suppose now $|C_i| = t$ for all i. Let $\mathcal{G}$ be the t-uniform hypergraph consisting of the C_i's as edges, each C_i with multiplicity $|\mathcal{F}_i|$. Then $|\mathcal{G}| = \sum_i |\mathcal{F}_i|$, $D(\mathcal{G}) \leq D$ and Lemma 2.2 implies that $\mathcal{G}$ is intersecting. If $\mathcal{G}$ contains no finite projective plane then Theorem 2.3 yields

$$|\mathcal{G}| \leq (t-1)D \ ,$$

which together with (2) gives

$$|\mathcal{H}| \leq \sum_i |\mathcal{F}_i| + f(k) = |\mathcal{G}| + f(k) \leq (t-1)D + f(k) \leq (t-1+1/t)D \ .$$

Suppose $\mathcal{G}$ contains a finite projective plane. Without loss of generality, we may and will assume that $C_1, \cdots, C_m$ form a projective plane, where $m = t^2 - t + 1$. Let A be the set of all m vertices in the projective plane, that is $A = \cup_{i=1}^m C_i$. Since every edge $E \in \mathcal{H}$ has a non-empty intersection with each C_i, we know that $|A \cap E| \geq t$. Thus

$$t|\mathcal{H}| \leq \sum_{E \in \mathcal{H}} |A \cap E| = \sum_{x \in A} d(x) \leq mD \ .$$

Therefore, we have

$$|\mathcal{H}| \leq (t-1+1/t)D \ ,$$

completing the proof.

$\square$

3 Concluding remarks and open problems

- It is easy to see that the assumption that $D(\mathcal{H})$ exceeds some function of k in Theorem 1.2 cannot be dropped. Indeed, for $k > 1$ the dual of a complete graph on $k+1$ vertices is a k-uniform, 1-simple intersecting hypergraph with maximum degree $D = 2$ and with $k+1$ ($>$ $(1-1+1/1) \cdot 2$) edges. Another example showing that $D(\mathcal{H})$ must exceed k in Theorem 1.2 is the set of lines in a projective plane of order $k-1$. This is a k-uniform, k-regular, 1-simple intersecting hypergraph with $k^2 - k + 1$ ($> (1-1+1/1) \cdot k$) edges. Moreover, it can be shown that in fact the assertion of Theorem 1.2 may fail unless $D(\mathcal{H})$ exceeds $2^{c\sqrt{k}}$ for some fixed absolute constant $c > 0$. Without trying to optimize the value of this constant, we prove this fact in the following proposition.

  Proposition 3.1 *For all sufficiently large k, there exists a k-uniform, t-simple intersecting hypergraph $\mathcal{H}'$ with m' edges and maximum degree D, where $\frac{m'}{3\sqrt{k}} \leq D < \frac{m'}{2\sqrt{k}}$, $t \leq \sqrt{k}$ and $m' \geq 0.08e^{\sqrt{k}/3}$.*

  Proof. The proof is probabilistic. To simplify the presentation we omit all floor and ceiling signs whenever these are not crucial. Let k be a large integer, define $l = 3\sqrt{k}$ and let $N_1, N_2, \ldots, N_k$ be k pairwise disjoint sets of cardinality l each. We construct a random k-uniform k-partite hypergraph on the set of vertices $N_1 \cup N_2 \cup \ldots \cup N_k$ as follows. Define $m = 0.1e^{k/l} = 0.1e^{\sqrt{k}/3}$ and let $\mathcal{H}$ be a collection of m edges, where each edge, randomly and independently, consists of k randomly chosen vertices, one from each N_i, all choices being equally probable. A pair of edges is called *good* if they have a non-empty intersection whose cardinality does not exceed $3k/l = \sqrt{k}$. Otherwise it is called *bad*. We claim that the probability that a fixed pair of edges is bad is smaller than $2e^{-k/l}$. To see this, note that the cardinality of the intersection of a pair is a Binomial random variable with parameters $p = 1/l$ and k. Thus, the probability that the intersection is empty is $(1 - 1/l)^k \leq e^{-k/l}$. The probability that the intersection contains more than $3k/l$ elements is at most $(e^2/3^3)^{k/l} < e^{-k/l}$, by the standard estimates for Binomial distributions; see, e.g., Theorem A.12 in Appendix A of [1]. Therefore, the expected number of bad pairs is at most $\binom{m}{2}2e^{-k/l} < 0.1m$. Hence, with probability at least a half the number of bad pairs is at most $0.2m$. The expected degree of a vertex in $\mathcal{H}$ is m/l and by the above mentioned estimates for Binomial distributions all degrees are at most $1.01m/l$ with probability (much) bigger than $1/2$. Thus there is a choice of the m edges as above so that there are less than $0.2m$ bad pairs and the maximum degree is at most $1.01m/l$. Fix such a choice, and let $\mathcal{H}'$ be the hypergraph obtained from $\mathcal{H}$ by throwing an edge from each bad pair. Then $\mathcal{H}'$ satisfies all the required properties in the assertion of the proposition. $\square$

- Jeff Kahn (private communication) suggested the following strong version of conjecture 1.1.

Conjecture 3.2 *For every $k \geq t \geq 1$ and every $\epsilon > 0$ there is a finite $D_0 = D_0(k, t, \epsilon)$ and a positive $\delta = \delta(k, t, \epsilon)$ so that every k-uniform hypergraph $\mathcal{H}$ with maximum degree at most D, where $D > D_0$, in which no set of $t + 1$ vertices is contained in more than δD edges satisfies*

$$\chi'(\mathcal{H}) \leq (t - 1 + 1/t + \epsilon)D.$$

For $t = 1$ and any k this holds, by the main result of [5], and it also holds for $k = t = 2$ by the general theorem of Vizing [7].

- Since, by definition, any hypergraph $\mathcal{H}$ contains a matching of size at least $|\mathcal{H}|/\chi'(\mathcal{H})$, the following conjecture is easier than Conjecture 1.1.

Conjecture 3.3 *For every $k \geq t \geq 1$ and every $\epsilon > 0$ there is a finite $D_0 = D_0(k, t, \epsilon)$ so that if $D > D_0$ then every k-uniform, t-simple hypergraph $\mathcal{H}$ with maximum degree at most D, contains a matching of size at least*

$$\frac{|\mathcal{H}|}{(t - 1 + 1/t + \epsilon)D}.$$

This is clearly the case for $k = t = 2$ and for $k > t = 1$, since the assertion of Conjecture 1.1 holds in these cases, and by the main result of Füredi in [3] the result is also correct for $t = k$; in fact, in [3] it is shown that in this case the ϵ term may be omitted. Any k-uniform hypergraph $\mathcal{H}$ (which is always k-simple as well) with maximum degree D, has a matching of size at least

$$\frac{|\mathcal{H}|}{(k-1+1/k)D}.$$

The general case of this last conjecture, however, remains open.

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