Asymmetric list sizes in bipartite graphs

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Abstract

Given a bipartite graph with parts A and B having maximum degrees at most Δ_A and Δ_B, respectively, consider a list assignment such that every vertex in A or B is given a list of colours of size k_A or k_B, respectively.

We prove some general sufficient conditions in terms of Δ_A, Δ_B, k_A, k_B to be guaranteed a proper colouring such that each vertex is coloured using only a colour from its list. These are asymptotically nearly sharp in the very asymmetric cases. We establish one sufficient condition in particular, where $\Delta_A = \Delta_B = \Delta$, $k_A = \log \Delta$ and $k_B = (1 + o(1))\Delta/\log \Delta$ as $\Delta \to \infty$. This amounts to partial progress towards a conjecture from 1998 of Krivelevich and the first author.

We also derive some necessary conditions through an intriguing connection between the complete case and the extremal size of approximate Steiner systems. We show that for complete bipartite graphs these conditions are asymptotically nearly sharp in a large part of the parameter space. This has provoked the following.

In the setup above, we conjecture that a proper list colouring is always guaranteed

- if $k_A \geq \Delta_A^\varepsilon$ and $k_B \geq \Delta_B^\varepsilon$ for any $\varepsilon > 0$ provided Δ_A and Δ_B are large enough;
- if $k_A \geq C \log \Delta_B$ and $k_B \geq C \log \Delta_A$ for some absolute constant $C > 1$; or
- if $\Delta_A = \Delta_B = \Delta$ and $k_B \geq C(\Delta/\log \Delta)^{1/k_A} \log \Delta$ for some absolute constant $C > 0$.

These are asymmetric generalisations of the above-mentioned conjecture of Krivelevich and the first author, and if true are close to best possible. Our general sufficient conditions provide partial progress towards these conjectures.

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1 Introduction

List colouring of graphs, whereby arbitrary restrictions on the possible colours used per vertex are imposed, was introduced independently by Erdős, Rubin and Taylor [10] and by Vizing [16]. Let $G = (V, E)$ be a simple, undirected graph. For a positive integer k, a mapping $L : V \to \binom{\mathbb{Z}^+}{k}$ is called a *k-list-assignment* of G; a colouring $c : V \to \mathbb{Z}^+$ is called an *L-colouring* if $c(v) \in L(v)$ for any $v \in V$. We say G is *k-choosable* if for any k-list-assignment L of G there is a proper L-colouring of G. The *choosability* $\mathrm{ch}(G)$ (or *choice number* or *list chromatic number*) of G is the least k such that G is k-choosable.

Note that if G is k-choosable then it is properly k-colourable. On the other hand, Erdős, Rubin and Taylor [10] exhibited bipartite graphs with arbitrarily large choosability. Let K_{n_1,n_2} denote a complete bipartite graph with part sizes n_1 and n_2.

Theorem 1 ([10])**.** *For some function $M(k)$ with $M(k) = 2^{k+o(k)}$ as $k \to \infty$,*

(i) $K_{n,n}$ is not k-choosable if $n \geq M(k)$, and

(ii) K_{n_1,n_2} is k-choosable if $n_1 + n_2 < M(k)$.

More succinctly Theorem 1 says that $\mathrm{ch}(K_{n,n}) \sim \log_2 n$ as $n \to \infty$.

There are two contrasting ways to try to strengthen this last statement. First does the lower bound hold more generally; that is, does a bipartite graph with minimum degree δ have choosability $\Omega(\log \delta)$ as $\delta \to \infty$? Indeed this was shown by the first author [1] using a probabilistic argument; later Saxton and Thomason [15] proved the asymptotically optimal lower bound of $(1 + o(1)) \log_2 \delta$. Second does the upper bound hold more generally; that is, does a bipartite graph with maximum degree Δ always have choosability $O(\log \Delta)$ as $\Delta \to \infty$? Krivelevich and the first author conjectured this in 1998.

Conjecture 2 ([2])**.** *There is some absolute constant $C > 0$ such that any bipartite graph of maximum degree at most Δ is k-choosable if $k \geq C \log \Delta$.*

Johansson's result for triangle-free graphs [12] gives the conclusion with $k \geq C\Delta/\log \Delta$, which is far from the conjectured bound. Conjecture 2 is an elegant problem in a well-studied area, but apart from constant factors[1] there has been no progress until now.

Despite the apparent difficulty of Conjecture 2, we propose an asymmetric refinement. Given a bipartite graph $G = (V{=}A \cup B, E)$ with parts A, B and positive integers k_A, k_B, a mapping $L : A \to \binom{\mathbb{Z}^+}{k_A}, B \to \binom{\mathbb{Z}^+}{k_B}$ is called a *(k_A, k_B)-list-assignment* of G. We say G is *(k_A, k_B)-choosable* if there is guaranteed a proper L-colouring of G for any such L.

Problem 3. *Given Δ_A and Δ_B, what are optimal choices of $k_A \leq \Delta_A$ and $k_B \leq \Delta_B$ for which any bipartite graph $G = (V{=}A \cup B, E)$ with parts A and B having maximum degrees at most Δ_A and Δ_B, respectively, is (k_A, k_B)-choosable?*

[1] In a recent advance, Molloy [14] considerably improved on the constant in Johansson's, cf. also [5].

We have the upper bounds on k_A, k_B, since the problem is trivial if $k_A > \Delta_A$ or $k_B > \Delta_B$.

Note that Problem 3, since it has a higher-dimensional parameter space, has wider scope than Conjecture 2 and is necessarily more difficult. However, the extra generality in Problem 3 has permitted a glimpse at an unexpected and basic connection between list colouring and combinatorial design theory (Theorem 6). This has prompted us to explore specific areas of the parameter space in Problem 3, motivating concrete versions of Problem 3 in the spirit of Conjecture 2 (Conjecture 7). One hope of ours is that further study of these problems may yield insights into Conjecture 2. But in fact, already in the present work, we have obtained asymmetric progress towards Conjecture 2 (Corollary 10).

Our first main result provides general progress towards Problem 3.

Theorem 4. *Let the positive integers Δ_A, Δ_B, k_A, k_B satisfy one of the following conditions, as stated or with roles exchanged between Δ_A and Δ_B and between k_A and k_B.*

(*i*) $k_B \ge (ek_A\Delta_B)^{1/k_A}\Delta_A$.

(*ii*) $e(\Delta_A(\Delta_B - 1) + 1)\left(1 - (1 - 1/k_B)^{\Delta_A \min\{1,k_B/k_A\}}\right)^{k_A} \le 1.$

Then any bipartite graph $G = (V{=}A \cup B, E)$ with parts A and B having maximum degrees at most Δ_A and Δ_B, respectively, is (k_A, k_B)-choosable.

Theorem 4 under condition (*i*) follows from a simple application of the Lovász Local Lemma, as we show in Section 2. This sufficient condition for Problem 3 is related to independent transversals in hypergraphs, cf. [8, 11], and to least conflict choosability [7]. In the most asymmetric settings, condition (*i*) is sharp up to a constant factor. We prove Theorem 4 under condition (*ii*) in Section 3 with the Lovász Local Lemma and a link to the coupon collector problem. In an attempt to clear up the 'parameter soup' arising from Problem 3 and the above sufficient conditions, we will discuss specific natural context for these after presenting some further results.

We complement our sufficient conditions for (k_A, k_B)-choosability with necessary ones, given mainly by the complete bipartite graphs. An easy boundary case is provided as follows, for warmup. This generalises a classic non-k-choosable construction.

Proposition 5. *For any $\delta, k \ge 2$, the complete bipartite graph $G = (V{=}A \cup B, E)$ with $|A| = \delta^k$ and $|B| = k$ is not (k, δ)-choosable.*

Proof. Let the vertices of B be assigned k disjoint lists of length δ, and let the vertices of A be assigned all possible k-tuples drawn from these k disjoint lists. □

This is best possible in the sense that the conclusion does not hold if $|A| < \delta^k$ or $|B| < k$; however, in Section 8 we exhibit a non-complete graph that is a slightly more efficient non-(k, δ)-choosable construction. On the other hand, Proposition 5 shows that condition (*i*) in Theorem 4 cannot be relaxed much in general. This follows for instance by considering the most asymmetric case, namely $k = k_A = \Delta_A$ and $\delta = \Delta_B^{1/k}$, with k fixed and $\Delta_B \to \infty$. Thus for fixed k_A and Δ_A Problem 3 is settled up to a constant factor.

Our second main result is a related but much broader necessary condition for Problem 3, also via the complete case. Let us write $\overline{M}(k_1, k_2, \ell)$ for the hypergraph Turán number defined as the minimum number of edges in a k_2-uniform hypergraph on ℓ vertices with no independent set of size $\ell - k_1$. This parameter is equivalent to the extremal size of approximate Steiner systems; in particular, $\overline{M}(k_1, k_2, \ell)$ is equal to the cardinality of a smallest k_1-$(\ell, \ell - k_2, \mathbb{Z}^+)$ design, cf. [9, Ch. 13] and [13]. We can draw a link between this classical extremal parameter and Problem 3. We show the following result in Section 4.

Theorem 6. *Let k_A, k_B, ℓ, k_1, k_2 be integers such that $k_A, k_B \geq 2$ and $\ell = k_1 + k_2 + 1$. The complete bipartite graph $G = (V{=}A \cup B, E)$ with $|A| = \overline{M}(k_1, k_A, \ell)$ and $|B| = \overline{M}(k_2, k_B, \ell)$ is not (k_A, k_B)-choosable.*

This link allows us, using known results for $\overline{M}$, to read off decent necessary conditions for specific parameterisations of Problem 3. As one example, a lower bound on $\mathrm{ch}(K_{n,n})$ of the form $\mathrm{ch}(K_{n,n}) \gtrsim \frac{1}{2}\log_2 n$ follows from Theorem 6 with the choice $k_1 = k_2 = k - 1$, $k_A = k_B = k$, and $\ell = 2k - 1$ for some k. As another example, a slightly weaker form of the necessary condition of Proposition 5 follows with the choice $k_1 = k_A(k_B - 1)$, $k_2 = k_A - 1$, and $\ell = k_A k_B$. We detail both of these easy examples in Section 4.

Note though that these last two examples show that Theorem 6 provides suboptimal necessary conditions for (k_A, k_B)-choosability even in the complete case. Furthermore, in Section 8 we give a construction to show that the complete case cannot, in general, be precisely extremal for Problem 3. Nevertheless, we surmise that Theorem 6 provides some good rough borders for Problem 3. More specifically, in Section 5 we give some basic sufficient conditions for (k_A, k_B)-choosability specific to the complete case and show in Section 6, through some routine asymptotic calculus, how these roughly match with Theorem 6 over a broad family of parameterisations for Problem 3. Then, just as Conjecture 2 was informed by Theorem 1, the asymptotic behaviour of (k_A, k_B)-choosability in the complete case leads us to conjecture the following concrete versions of Problem 3.

Conjecture 7. *Let the positive integers Δ_A, Δ_B, k_A, k_B satisfy one of the following.*

(i) Given $\varepsilon > 0$, we have $\Delta_A, \Delta_B \geq \Delta_0$ for some $\Delta = \Delta_0(\varepsilon)$, and

$$k_A \geq \Delta_A^{\varepsilon} \quad \textit{and} \quad k_B \geq \Delta_B^{\varepsilon}.$$

(ii) For some absolute constant $C > 1$,

$$k_A \geq C \log \Delta_B \quad \textit{and} \quad k_B \geq C \log \Delta_A.$$

(iii) $\Delta_A = \Delta_B = \Delta$, and, for some absolute constant $C > 0$,

$$k_B \geq C(\Delta/\log\Delta)^{1/k_A} \log \Delta \quad \textit{or} \quad k_A \geq C(\Delta/\log\Delta)^{1/k_B} \log \Delta.$$

Then any bipartite graph $G = (V{=}A \cup B, E)$ with parts A and B having maximum degrees at most Δ_A and Δ_B, respectively, is (k_A, k_B)-choosable.

Conjecture 7 constitutes three natural asymmetric analogues of Conjecture 2. The first is weaker than Conjecture 2; the latter two are stronger. We discuss these separately in turn.

Behind condition (i), Conjecture 7 concerns the question, for what positive functions f does $k_A \ge f(\Delta_A)$ and $k_B \ge f(\Delta_B)$ suffice for (k_A, k_B)-choosability, no matter how far apart Δ_A and Δ_B are? Essentially we posit that some f with $f(x) = x^{o(1)}$ as $x \to \infty$ will work, and this would be best possible due to the complete case (Theorem 16(i)). In particular, $f(x) = O(\log x)$ is impossible, in contrast to Conjecture 2. But in fact, Theorem 4 shows that Conjecture 7 under condition (i) reduces to its most symmetric form.

Corollary 8. *Given $0 < \varepsilon < 1$ and integers Δ_A, Δ_B satisfying $\Delta_B > \Delta_A^{2/\varepsilon}$ and $\Delta_A > (4/\varepsilon)^{1/\varepsilon}$, any bipartite graph $G = (V{=}A \cup B, E)$ with parts A and B having maximum degrees at most Δ_A and Δ_B, respectively, is $(\Delta_A^\varepsilon, \Delta_B^\varepsilon)$-choosable.*

As a consequence, Conjecture 7 under condition (i) follows from the same assertion under the further assumption that $\Delta_A = \Delta_B = \Delta$.

Proof. It suffices to check condition (i) of Theorem 4 with $k_A = \Delta_A^\varepsilon$ and $k_B = \Delta_B^\varepsilon$. Indeed, as $\Delta_B^2 > e\Delta_A^\varepsilon\Delta_B$, $\varepsilon/4 > 1/\Delta_A^\varepsilon$ and $\Delta_B^{\varepsilon/2} > \Delta_A$, we have $\Delta_B^\varepsilon > (e\Delta_A^\varepsilon\Delta_B)^{1/\Delta_A^\varepsilon}\,\Delta_A$.

Assume condition (i) of Conjecture 7, and moreover assume the truth of the conjecture only for its most asymmetric form $\Delta_A = \Delta_B = \Delta$. We may assume that $\Delta_0 > (4/\varepsilon)^{1/\varepsilon}$, and so, without loss of generality, we may also assume by the first part that $\Delta_A \le \Delta_B \le \Delta_A^{2/\varepsilon}$. For any bipartite graph $G = (V{=}A \cup B, E)$ with parts A and B having maximum degrees at most Δ_A and Δ_B, we have from the assumption that G is k-choosable for, say, $k = \Delta_B^{\varepsilon^3} \le \Delta_A^{2\varepsilon^2}$, provided Δ_B, and thus Δ_A, is large enough as a function of ε. This is smaller than Δ_A^ε for Δ_A sufficiently large as a function of ε, as required. □

Thus Conjecture 7 under condition (i) would follow from a weaker form of Conjecture 2. By an earlier work due to Davies, de Joannis de Verclos, Pirot and the third author [5], we so far know that $f(x) = (1 + o(1))x/\log x$ works.

Under condition (ii), Conjecture 7 concerns a 'crossed' version of the previous question, so for what positive functions g does $k_A \ge g(\Delta_B)$ and $k_B \ge g(\Delta_A)$ suffice for (k_A, k_B)-choosability? In this case, we conjecture that $g(x) = O(\log x)$ will work, which coincides with Conjecture 2 in the symmetric case $\Delta_A = \Delta_B = \Delta$. The complete graphs demonstrate the hypothetical sharpness of this assertion up to a constant factor for nearly the entire range of possibilities for Δ_A and Δ_B (Theorem 16(ii)). Some modest partial progress towards Conjecture 7 under condition (ii) follows from Theorem 4.

Corollary 9. *Given $\varepsilon > 0$, there exists δ such that for Δ_B large enough, any bipartite graph $G = (V{=}A \cup B, E)$ with parts A and B having maximum degrees at most Δ_A and Δ_B, respectively, is $(\delta \log \Delta_B, (1+\varepsilon)\Delta_A)$- and $(\log \Delta_B, (e+\varepsilon)\Delta_A)$-choosable.*

Proof. This follows for Δ_B large enough from condition (i) of Theorem 4 with either $k_A = \delta \log \Delta_B$ and $k_B = (1+\varepsilon)\Delta_A$ or with $k_A = \log \Delta_B$ and $k_B = (e+\varepsilon)\Delta_A$. □

Under condition (iii), Conjecture 7 concerns the setting most closely related to Conjecture 2. It suggests how Problem 3 might behave for the symmetric case $\Delta_A = \Delta_B = \Delta$.

The complete graphs demonstrate its hypothetical sharpness up to a constant factor for the entire range of possibilities for k_A, and thus symmetrically k_B (Theorem 16(iii)). Theorem 4 provides the following partial progress towards Conjecture 7 under condition (iii). In fact, this constitutes significant (asymmetric) progress towards Conjecture 2, and is the first substantial advance in this longstanding problem.

Corollary 10. *Given $\varepsilon > 0$, any bipartite graph $G = (A \cup B, E)$ with parts A and B having maximum degree at most Δ is $((1+\varepsilon)\Delta/\log_4 \Delta, 2)$- and $((1+\varepsilon)\Delta/\log \Delta, \log \Delta)$-choosable for all Δ large enough.*

Proof. This follows for Δ large enough from condition (ii) in Theorem 4 with $\Delta_A = \Delta_B = \Delta$ and either $k_A = (1+\varepsilon)\Delta/\log_4 \Delta$ and $k_B = 2$ or $k_A = (1+\varepsilon)\Delta/\log \Delta$ and $k_B = \log \Delta$. □

In Section 7 we are able to nearly completely characterise (k_A, k_B)-choosability for complete bipartite graphs if $k_A \geq \Delta_A - 1$. In Section 9, we give a connection between minimum degree and (k_A, k_B)-choosability along the lines of [1].

Probabilistic preliminaries

We will use the following standard probabilistic tools, cf. [3].

A Chernoff Bound. *For $X \sim \mathrm{Bin}(n,p)$ and $\varepsilon \in [0,1]$, $\mathbb{P}(X < (1-\varepsilon)np) \leq \exp(-\frac{\varepsilon^2}{2}np)$.*

The Lovász Local Lemma. *Consider a set $\mathcal{E}$ of (bad) events such that for each $A \in \mathcal{E}$*

(i) *$\mathbb{P}(A) \leq p < 1$, and*

(ii) *A is mutually independent of a set of all but at most d of the other events.*

If $ep(d+1) \leq 1$, then with positive probability none of the events in $\mathcal{E}$ occur.

2 A sufficient condition via transversals

In this section, we prove Theorem 4 under condition (i). We prefer to state and prove a slightly stronger form. To do so, we need some notational setup. Let $H = (V, E)$ be a hypergraph. The *degree* of a vertex in H is the number of edges containing it. Given some partition of V, a *transversal* of H is a subset of V that intersects each part in exactly one vertex. A transversal of H is called *independent* if it contains no edge, cf. [8].

Lemma 11. *Fix $k \geq 2$. Let H be a k-uniform vertex-partitioned hypergraph, each part being of size ℓ, such that every part has degree sum at most Δ. If $\ell^k \geq e(k(\Delta-1)+1)$, then H has an independent transversal.*

Let us first show Lemma 11 implies Theorem 4 under condition (i).

Proof of Theorem 4 under condition (i). Let $k_A, k_B, \Delta_A, \Delta_B$ satisfy condition (i). Let L be a (k_A, k_B)-list-assignment of G. We would like to show that there is a proper L-colouring of G. We do so by defining a suitable hypergraph $H = (V_H, E_H)$.

Let (w, c) be a vertex of V_H if $w \in B$ and $c \in L(w)$.

Let $((w_1, c_1), \ldots, (w_{k_A}, c_{k_A}))$ be an edge of E_H whenever there is some $v \in A$ such that $N(v) \supseteq \{w_1, \ldots, w_{k_A}\}$ and $L(v) = \{c_1, \ldots, c_{k_A}\}$.

Note that H is a k_A-uniform vertex-partitioned hypergraph, where the parts are naturally induced by each list in B and so are each of size k_B. We have defined H and its partition so that any independent transversal corresponds to a special partial L-colouring of G. In particular, it is an L-colouring of the vertices in B for which there is guaranteed to be a colour in $L(v)$ available for every $v \in A$, and so it can be extended to a proper L-colouring of all G.

Every part in H has degree sum at most $\Delta_B \binom{\Delta_A - 1}{k_A - 1} k_A! \le \Delta_B \Delta_A^{k_A}$, so the result follows from Lemma 11 with $\ell = k_B$ and $\Delta = \Delta_B \Delta_A^{k_A}$. $\square$

Proof of Lemma 11. Write $H = (V, E)$ and suppose $\ell^k \ge e(k(\Delta - 1) + 1)$. Consider the random transversal $\mathbf{T}$ formed by choosing one vertex from each part independently and uniformly. For each edge $f \in E$, let A_f denote the event that $\mathbf{T} \supseteq f$. Note that $\mathbb{P}(A_f) \le 1/\ell^k$. Moreover A_f is mutually independent of a set of all but at most $k(\Delta - 1)$ of the other events $A_{f'}$. The transversal $\mathbf{T}$ is independent if none of the events A_f occur. Since by assumption $e(1/\ell^k)(k(\Delta - 1) + 1) \le 1$, there is a positive probability that $\mathbf{T}$ is independent by the Lovász Local Lemma. $\square$

3 A sufficient condition via coupon collection

Proof of Theorem 4 under condition (ii). Let $k_A, k_B, \Delta_A, \Delta_B$ satisfy condition (ii). Let L be a (k_A, k_B)-list-assignment of G. We would like to show that there is a proper L-colouring of G. To this end, colour each vertex $w \in B$, randomly and independently, by a colour chosen uniformly from its list $L(w)$. Let $T_{v,c}$ be the event that $v \in A$ has a neighbour coloured with colour c. Let T_v be the event that $T_{v,c}$ happens for all $c \in L(v)$. The proof hinges on the following claim, which is related to the coupon collector problem, cf. e.g. [6].

Claim. *The events $T_{v,c}$, for fixed v as c ranges over all colours in $L(v)$, are negatively correlated. In particular, $\mathbb{P}(T_v) \le \prod_{c \in L(v)} \mathbb{P}(T_{v,c})$.*

Proof. We have to prove, for every $I \subset L(v)$, that $\mathbb{P}(\forall c \in I \colon T_{v,c}) \le \prod_{c \in I} \mathbb{P}(T_{v,c})$. If some colour $c \in L(v)$ is not in the list of any neighbour of v, both sides equal zero and so the inequality holds. So assume this is not the case. We prove the statement by induction on $|I|$. When $|I| \le 1$ the statement is trivially true. Let $I \subset L(v)$ be a subset for which the statement is true and let $c' \in L(v) \setminus I$. We now prove the statement for $I' = I \cup \{c'\}$. We have $\mathbb{P}(\forall c \in I \colon T_{v,c}) \le \mathbb{P}(\forall c \in I \colon T_{v,c} \mid \neg T_{v,c'})$ as the probability to use a colour in I is larger if in all neighbouring lists the colour c' is removed. This is equivalent to

$$\begin{aligned} &\mathbb{P}(\forall c \in I\colon T_{v,c}) \geq \mathbb{P}(\forall c \in I\colon T_{v,c} \mid T_{v,c'}) \\ \iff &\mathbb{P}(\forall c \in I'\colon T_{v,c}) \leq \mathbb{P}(\forall c \in I\colon T_{v,c})\,\mathbb{P}(T_{v,c'}) \end{aligned}$$

This last expression is at most $\prod_{c\in I'} \mathbb{P}(T_{v,c})$ by the induction hypothesis, as desired. $\Diamond$

For the i^{th} colour c_i in $L(v)$, let the number of occurrences of c_i in the neighbouring lists of v be x_i. Note that $\mathbb{P}(T_{v,c_i}) = 1-(1-1/k_B)^{x_i}$. Using $x_i \leq \Delta_A$ for every $1 \leq i \leq k_A$ and the claim, we have

$$\mathbb{P}(T_v) \leq \left(1-\left(1-\frac{1}{k_B}\right)^{\Delta_A}\right)^{k_A}.$$

Noting that $\sum_{i=1}^{k_A} x_i \leq k_B\Delta_A$ and that the function $\log(1-(1-1/k_B)^x)$ is concave and increasing, Jensen's Inequality applied with the claim implies that

$$\mathbb{P}(T_v) \leq \left(1-\left(1-\frac{1}{k_B}\right)^{k_B\Delta_A/k_A}\right)^{k_A}.$$

Each event T_v is mutually independent of all other events T_u apart from those corresponding to vertices $u \in A$ that have a common neighbour with v in G. As there are at most $\Delta_A(\Delta_B-1)$ such vertices besides v, the Lovász Local Lemma guarantees with positive probability that none of the events T_v occur, i.e. there is a proper L-colouring, as desired. $\square$

4 The complete case and Steiner systems

In this section, we investigate general necessary conditions for (k_A, k_B)-choosability via the complete bipartite graphs. Inspired in part by Theorem 1 and related work of Bonamy and the third author [4], this leads naturally to the study of an extremal set theoretic parameter. For positive integers k_1, k_2, ℓ, we say that a family $\mathcal{F}$ of k_2-element subsets of $[\ell]$ has *Property $A(k_1, k_2, \ell)$* (A is for asymmetric) if there is a k_1-element subset of $[\ell]$ that intersects every set in $\mathcal{F}$. We then define $\overline{M}(k_1, k_2, \ell)$ to be the cardinality of a smallest family of k_2-element subsets of $[\ell]$ that does not have Property A(k_1, k_2, ℓ). Note that this definition of $\overline{M}$ coincides with the definition given before the statement of Theorem 6.

Proof of Theorem 6. We define a (k_A, k_B)-list-assignment L as follows. Let $\mathcal{F}_1$ be a family of $\overline{M}(k_1, k_A, \ell)$ k_A-element subsets of $[\ell]$ without Property A(k_1, k_A, ℓ). Let $\mathcal{F}_2$ be a family of $\overline{M}(k_2, k_B, \ell)$ k_B-element subsets of $[\ell]$ without Property A(k_2, k_B, ℓ). Assign the sets of $\mathcal{F}_1$ as lists to the vertices in A and the sets of $\mathcal{F}_2$ as lists to the vertices in B. Suppose that c is an L-colouring. Then $C_1 = \{c(a) \mid a \in A\}$ intersects every set in $\mathcal{F}_1$ and so $|C_1| \geq k_1+1$ by assumption and similarly $C_2 = \{c(b) \mid b \in B\}$ has cardinality $|C_2| \geq k_2+1$. So $|C_1|+|C_2| \geq k_1+k_2+2 > \ell$, implying that c cannot be a proper colouring. $\square$

Since $\overline{M}(\ell-k_A, k_A, \ell) = \binom{\ell}{k_A}$, we have the following corollary of Theorem 6.

Corollary 12. *Let k_A, k_B, ℓ be integers such that $k_A, k_B \geq 2$ and $\ell \geq k_A + k_B - 1$. The complete bipartite graph $G = (V{=}A \cup B, E)$ with $|A| = \binom{\ell}{k_A}$ and $|B| = \overline{M}(k_A - 1, k_B, \ell)$ is not (k_A, k_B)-choosable.*

Although they are trivial, the following observations already show that Theorem 6 and Corollary 12 cannot be improved much in general.

- If $k_A = k_B = k$ and $\ell = 2k - 1$, then $\overline{M}(k_A - 1, k_B, \ell) = \binom{2k-1}{k} = 2^{2k+o(k)}$ and Corollary 12 implies that $K_{\binom{2k-1}{k},\binom{2k-1}{k}}$ is not k-choosable. This implies $\mathrm{ch}(K_{n,n}) \gtrsim \frac{1}{2}\log_2 n$, which one can compare to the bounds of Theorem 1.
- If $\ell = k_A \cdot k_B$, then $\overline{M}(k_A - 1, k_B, \ell) = k_A$ and so Corollary 12 implies that $K_{\binom{k_A \cdot k_B}{k_A},k_A}$ is not (k_A, k_B)-choosable. This produces a necessary condition on k_B for (k_A, k_B)-choosability only slightly weaker than Proposition 5.

We present several more instances where Theorem 6 and Corollary 12 are nearly sharp in Section 6. For this, we will have use for the following estimates for the parameter $\overline{M}$. A version of this result can be found in [9, Ch. 13], but for completeness, we present a standard derivation in the appendix.

Theorem 13. *Let k_1, k_2, ℓ be integers such that $k_1, k_2 \geq 2$ and $\ell \geq k_1 + k_2$. Then*

$$\frac{\ell!(\ell - k_1 - k_2)!}{(\ell - k_2)!(\ell - k_1)!} \leq \overline{M}(k_1, k_2, \ell) < \frac{\ell!(\ell - k_1 - k_2)!}{(\ell - k_2)!(\ell - k_1)!} \log \binom{\ell}{k_1}.$$

5 Sufficient conditions in the complete case

In this section, we give general sufficient conditions for (k_A, k_B)-choosability of complete bipartite graphs $K_{a,b}$. Our strategy for establishing (k_A, k_B)-choosability in this setting is to take a random bipartition of the set of all colours and try to use one part for colouring A and the other part for colouring B. This yields the following lemma.

Lemma 14. *Let the reals $0 < \varepsilon, p < 1$ and positive integers a, b, k_A, k_B satisfy either*

$$ap^{k_A} + b(1 - p)^{k_B} < 1 \text{ or} \tag{1}$$

$$\frac{ap^{k_A - 1}}{(1 - \varepsilon)k_B} + b \exp\left(-\varepsilon^2 k_B p/2\right) < 1. \tag{2}$$

Then the complete bipartite graph $G = (V{=}A \cup B, E)$ with $|A| = a$ and $|B| = b$ is (k_A, k_B)-choosable.

Proof. Let L be (k_A, k_B)-list-assignment of G. Let $U = \cup_{v \in V} L(v)$, i.e. U is the union of all colour lists. Define a random partition of U into parts L_A and L_B as follows. For each colour in U, randomly and independently assign it to L_B with probability p and otherwise assign it to L_A.

First assume (1). For a given vertex $v \in A$, the probability that $L(v) \cap L_A = \emptyset$ is p^{k_A}. For a given vertex $v \in B$, the probability that $L(v) \cap L_B = \emptyset$ is $(1-p)^{k_B}$. So the expected number of vertices v which cannot be coloured with the corresponding list L_A or L_B is equal to $ap^{k_A} + b(1-p)^{k_B}$. So the probabilistic method then guarantees a fixed choice of the parts L_A and L_B such that every vertex in A can be coloured with a colour from L_A and every vertex in B with a colour from L_B.

Otherwise assume (2). For a given vertex $w \in B$, the random variable $|L(w) \cap L_B|$ has a binomial distribution of parameters k_B and p. So by a Chernoff bound the probability that $|L(w) \cap L_B|$ is smaller than $(1-\varepsilon)k_B p$ is smaller than $\exp(-\varepsilon^2 k_B p/2)$. Thus, with probability greater than $1 - b\exp(-\varepsilon^2 k_B p/2)$, we have $|L(w) \cap L_B| \geq (1-\varepsilon)k_B p$ for all $w \in B$.

For a given vertex $v \in A$, the probability that $L(v) \cap L_A = \emptyset$ is p^{k_A}. So the expected number of vertices v for which this holds is $p^{k_A}|A|$. Thus by Markov's inequality, the probability that there are at least $(1-\varepsilon)k_B p$ such vertices v is smaller than $\frac{p^{k_A-1}a}{(1-\varepsilon)k_B}$.

Due to (2), the probabilistic method guarantees a fixed choice of partition of U into parts L_A and L_B such that $|L(w) \cap L_B| \geq (1-\varepsilon)k_B p$ for every $w \in B$ and the number of vertices $v \in A$ such that $L(v) \cap L_A = \emptyset$ is smaller than $(1-\varepsilon)k_B p$. Colour any vertex $v \in A$ having $L(v) \cap L_A = \emptyset$ with an arbitrary colour from $L(v)$. Colour any other vertex $v' \in A$ with a colour from $L(v') \cap L_A$. Finally colour any vertex $w \in B$ with some colour in $L(w) \cap L_B$ unused by any vertex of A — this is possible as there were fewer than $(1-\varepsilon)k_B p$ colours of L_B used to colour vertices in A and the lists in B all have at least $(1-\varepsilon)k_B p$ colours from L_B.

In either case, we are guaranteed a proper L-colouring of G, as promised. □

6 Asymptotic sharpness in the complete case

We next use the results of Sections 4 and 5 to roughly settle the behaviour of complete bipartite graphs with respect to Problem 3 in several regimes. The conditions in Theorems 15 and 16 naturally correspond to the conditions in Conjecture 7.

Theorem 15. *Let the positive integers a, b, k_A, k_B satisfy one of the following.*

(i) For any $\varepsilon > 0$, $a, b \geq \Delta_0$ for some $\Delta_0 = \Delta_0(\varepsilon)$, $k_A > b^\varepsilon$ and $k_B > a^\varepsilon$.

(ii) For any $t > 0$, $k_A \geq \frac{1}{t}\log_2(2a)$ and $k_B > 2^t \log(2b)$.

(iii) We have that $a = b = \Delta$ and either $k_B > 8(\Delta/(2\log(2\Delta)))^{1/k_A}\log(2\Delta)$ or $k_A > 8(\Delta/(2\log(2\Delta)))^{1/k_B}\log(2\Delta)$.

Then the complete bipartite graph $G = (V{=}A \cup B, E)$ with $|A| = a$ and $|B| = b$ is (k_A, k_B)-choosable.

Proof. Assume condition (i) and assume without loss of generality that $a \geq b$. Fix $\varepsilon > 0$ and take $\Delta_0 > 2$ large enough such that $\varepsilon\Delta_0^\varepsilon > 3$ and $x^{\varepsilon/3} > \log(2x)$ for every $x \geq \Delta_0$.

Now $k_B \geq a^{\varepsilon} > a^{2/b^{\varepsilon}} a^{\varepsilon/3} > (2a)^{1/b^{\varepsilon}} \log(2b)$. By taking $p = (2a)^{-1/b^{\varepsilon}}$, we have that (1) is satisfied and the result follows from Lemma 14.

Assume condition (ii). Take $p = 1/2^t$. Then $ap^{k_A} \leq 1/2$ and $b(1-p)^{k_B} \leq b\exp(-pk_B) < 1/2$, so that (1) is satisfied and the result follows from Lemma 14.

Assume condition (iii) and by symmetry assume $k_B > 8(\Delta/(2\log(2\Delta)))^{1/k_A}\log(2\Delta)$. Take $p = 8\log(2\Delta)/k_B$ and $\varepsilon = 1/2$. Then $p < (2\log(2\Delta)/\Delta)^{1/k_A}$ and $b\exp\left(-\varepsilon^2 k_B p/2\right) < 1/2$. Also

$$\frac{ap^{k_A}}{(1-\varepsilon)pk_B} = \frac{\Delta p^{k_A}}{4\log(2\Delta)} < \frac{1}{2}.$$

Thus (2) is satisfied and the result follows from Lemma 14. □

The conditions are also sharp in some sense.

Theorem 16. *For each of the following four conditions, there are infinitely many choices of the positive integers a, b, k_A, k_B satisfying it such that the complete bipartite graph $G = (V{=}A \cup B, E)$ with $|A| = a$ and $|B| = b$ is not (k_A, k_B)-choosable.*

(i) *Given a monotone real function g satisfying $g(1) \geq 1$ and $g(x) = \omega(1)$ as $x \to \infty$,*

$$k_A > b^{1/g(b)} \text{ and } k_B = a.$$

(ii) *For any integer $t \geq 4$, either*

$$k_A \leq \frac{\log_2 a}{t} \text{ and } k_B \leq \frac{2^{t-1}}{e}(\log b - \log\log a) \leq \frac{\log_2 a}{t},$$

or

$$k_A \leq 2^t \log_2 a \text{ and } k_B \leq \frac{\log_2 b - \log_2 \log a}{t + \log_2 t + 3},$$

where $4 \leq b \leq a$.

(iii) *For any fixed integer $k > 1$ and Δ sufficiently large, $\Delta_A = \Delta_B = \Delta$, $k_A = k$ and $k_B = c(\Delta/\log\Delta)^{1/k}\log\Delta$ for some constant $c = c(k)$.*

Proof. Consider condition (i). Take δ large enough so that $g(\delta^a) > a$. Let $k_A = \delta$ and $b = \delta^a$ and note that $b^{1/g(b)} = \delta^{a/g(b)} < \delta = k_A$, i.e. $k_A^{g(b)} > b$. The result then follows from Proposition 5.

Consider condition (ii). For the former case, let

$$\ell = \frac{2^t}{et}\log_2 a \quad \text{and} \quad k_A = \frac{\log_2 a}{t} \geq k_B = \frac{2^{t-1}}{e}(\log b - \log\log a).$$

Then $\binom{\ell}{k_A} \le (e\ell/k_A)^{k_A} = a$. Also

$$\binom{\ell}{k_B} \Big/ \binom{\ell - k_A + 1}{k_B} \le \left(\frac{\ell - k_B}{\ell - k_A - k_B}\right)^{k_B} \le \left(\frac{2^t - e}{2^t - 2e}\right)^{k_B} \le \exp\left(\frac{2e}{2^t} k_B\right) = \frac{b}{\log a}.$$

Here we used the estimation $(x - e)/(x - 2e) \le \exp(2e/x)$ for $x \ge 16$.

For the latter case, choose

$$\ell = \left(2^t + \frac{1}{t + \log_2 t + 3}\right) \log_2 a, \quad k_A = 2^t \log_2 a \quad \text{and} \quad k_B = \frac{\log_2 b - \log_2 \log a}{t + \log_2 t + 3}.$$

Then $\binom{\ell}{k_A} = \binom{\ell}{\ell - k_A} \le (e\ell/(\ell - k_A))^{\ell - k_A} \le a$ since

$$e\ell/(\ell - k_A) \le e(2^t(t + \log_2 t + 3) + 1) \le 8t2^t = 2^{t + \log_2 t + 3}.$$

Also

$$\binom{\ell}{k_B} \Big/ \binom{\ell - k_A + 1}{k_B} \le \left(\frac{e\ell}{\ell - k_A}\right)^{k_B} \le \frac{b}{\log a}.$$

In either case, the result follows from Corollary 12 and Theorem 13.

Consider condition (iii). First we make a computation verifying $\overline{M}((\log m)/2, m/2, m) < m$ when m is sufficiently large. By Theorem 13, for m sufficiently large, we have that

$$\overline{M}((\log m)/2, m/2, m) \le \left(\frac{m}{m/2 - (\log m)/2}\right)^{(\log m)/2} \log \binom{m}{(\log m)/2} < \sqrt{m}(\log m)^2 < m.$$

We choose $m = c(\Delta/\log \Delta)^{1/k} \log \Delta$, where $c = 1/(4k(k-1))$. By the above computation, this choice satisfies $\overline{M}((\log m)/2, m/2, m) \le \Delta$ for Δ large enough. Let b be such that $(k-1)b \log \Delta = (\log m)/2$, and note that $b \sim 2c$ as $\Delta \to \infty$.

Let $G = (A \cup B, E)$ be a complete bipartite graph with

$$|A| = \overline{M}((\log m)/2, m/2, m) \quad \text{and} \quad |B| = b \log \Delta \binom{\lceil m/(b \log \Delta) \rceil}{k}.$$

Note that as $\Delta \to \infty$

$$|B| \le b \log \Delta \frac{\lceil m/(b \log \Delta) \rceil^k}{k!} \sim \frac{1}{k!} \left(\frac{c}{b}\right)^k b\Delta \le \Delta$$

and thus $\Delta_A, \Delta_B \le \Delta$ for all Δ large enough.

We define a (k_A, k_B)-list-assignment L as follows. Let $\mathcal{F}$ be a family of $|A|$ $(m/2)$-element subsets of $[m]$ without Property A$((\log m)/2, m/2, m)$ and assign the sets of $\mathcal{F}$ as lists to the vertices in A. So there is no $((\log m)/2)$-element subset of $[m]$ that intersects every list in A. For B, arbitrarily partition $[m]$ into $b \log \Delta$ segments of nearly equal size, and assign as lists to the vertices in B all possible k-element subsets chosen from within a single segment.

Note that for any L-colouring, $\cup_{w \in B} L(w)$ intersects at least one colour from each k-element subset of a segment, and so $\cup_{w \in B} L(w)$ avoids at most $(k-1)b \log \Delta = (\log m)/2$ colours of $[m]$. However, as noted above, $\cup_{v \in A} L(v)$ must have more than $(\log m)/2$ colours of $[m]$, and this precludes a proper L-colouring. □

7 Sharpness in a boundary complete case

In this section, we precisely solve Problem 3 for complete bipartite graphs when $k_A \geq \Delta_A - 1$. When $k_A \geq \Delta_A + 1$, we know that any bipartite $G = (A \cup B, E)$ must be (k_A, k_B)-choosable. The case $k_A = \Delta_A$ is handled by Proposition 5 and the fact that its conclusion fails if $|A| < \delta^k$ or $|B| < k$. The remainder of this section is devoted to the case $k_A = \Delta_A - 1$.

Proposition 17. *Let $b \geq 5$ and $\delta = q(b-1) + r$ with $0 \leq r \leq b-2$ and q being integers such that $\delta \gg b$. Then the complete bipartite graph $G = (V{=}A \cup B, E)$ with $|A| = a$ and $|B| = b$ is $(b-1, \delta)$-choosable if and only if*

$$a < \delta^{b-1} - ((b-2)q + r)^{b-1-r}((b-2)q + r - 1)^r.$$

Note that for $1 \ll b \ll \delta$, this bound on a is approximately $(1 - 1/e)\delta^{b-1}$.

Proof. We write $B = \{v_1, v_2, \ldots, v_b\}$. Let L be a $(b-1, \delta)$-list-assignment of G. If some colour appears in the list of at least three different vertices of B, or two colours appear in common to the lists of two disjoint pairs of vertices of B, then we can certainly L-colour G. Let $L_{12} = L(v_1) \cap L(v_2)$ and analogously define L_{ij} for each $1 \leq i < j \leq b$ and let $\ell_{ij} = |L_{ij}|$. We know the index set of the non-empty L_{ij} form an intersecting family, so there are two cases depending on this family being trivial or non-trivial. We consider these cases in turn.

(*i*) (non-trivial family) Without loss of generality, we may assume that only L_{12}, L_{13} and L_{23} are non-empty among all L_{ij}. If there are $b-1$ colours which do not appear in the list of a vertex of A, such that the vertices of B can coloured with these $b-1$ colours, then G is L-colourable. So if G is not L-colourable, every such collection of $b-1$ colours has to appear as a list of a vertex in A. As there are $(\delta - \ell_{12} - \ell_{13})(\delta - \ell_{12} - \ell_{23})(\delta - \ell_{13} - \ell_{23})\delta^{b-3}$ combinations of b colours appearing only once among the lists of the vertices in B, every $(b-1)$-list can forbid at most δ of these. This implies that G is L-colourable if

$$\begin{aligned} a &< (\ell_{12} + \ell_{13} + \ell_{23})\delta^{b-2} - (\ell_{12}\ell_{13} + \ell_{12}\ell_{23} + \ell_{13}\ell_{23})\delta^{b-3} \\ &\quad + (\delta - \ell_{12} - \ell_{13})(\delta - \ell_{12} - \ell_{23})(\delta - \ell_{13} - \ell_{23})\delta^{b-4} \\ &= \delta^{b-1} - (\ell_{12} + \ell_{13} + \ell_{23})\delta^{b-2} + (\ell_{12} + \ell_{13} + \ell_{23})^2\delta^{b-3} \\ &\quad - (\ell_{12} + \ell_{13})(\ell_{12} + \ell_{23})(\ell_{13} + \ell_{23})\delta^{b-4}. \end{aligned}$$

The minimum occurs for $\ell_{12} = \ell_{23} = \ell_{13} = \delta/4$, leading to a bound of $\frac{11}{16}\delta^{b-1}$.

(*ii*) (trivial family) Without loss of generality, we assume only some of the L_{1j} are non-empty and $\ell_{1b} = \min\{\ell_{1j} \mid 2 \leq j \leq b\}$. The number of choices of a set S of $b-1$ colours such that any vertex has at least one colour of S in its list equals

$$\delta^{b-1} - \prod_{j=2}^{b}(\delta - \ell_{1j}).$$

We also need to forbid colouring the vertices of B with all colours that appear only once in a list in B. For this we need to forbid at least an additional

$$\left(\delta - \sum_{j=2}^{b} \ell_{1j}\right) \prod_{j=2}^{b-1} (\delta - \ell_{1j})$$

sets S with $b-1$ elements. The total number of sets to forbid equals

$$\delta^{b-1} - \left(\sum_{j=2}^{b-1} \ell_{1j}\right) \prod_{j=2}^{b-1} (\delta - \ell_{1j}).$$

This expression is minimised when $\delta - \sum_{j=2}^{b-1} \ell_{1j}$ and every ℓ_{1j} for $2 \le j \le b-1$ are equal to a number of the form $\lfloor \frac{\delta}{b-1} \rceil$. When $b \ll \delta$ and $b \ge 5$, we have that this is approximately $(1-(1-1/(b-1))^{b-1})\delta^{b-1}$, which is smaller than the value $\frac{11}{16}\delta^{b-1}$ obtained in the previous case. Equality can be attained, i.e. G is not $(b-1,\delta)$-choosable when $a = \delta^{b-1} - ((b-2)q+r)^{b-1-r}((b-2)q+r-1)^r$ (or larger) as we can take the lists being exactly those mentioned before for minimising the expression. □

The same analysis also gives the result for $b \in \{3,4\}$. When $b=3$, the bound for a is $\lfloor \frac{3}{4}\delta^2 \rceil$. When $b=4$, the same analysis as in Proposition 17 gives that the bound for a occurs in the first case (non-trivial family), resulting in the following detailed proposition.

Proposition 18. *Let $\delta \ge 2$. The complete bipartite graph $G = (V{=}A \cup B, E)$ with $|A| = a$ and $|B| = 4$ is $(3,\delta)$-choosable if and only if*

$$a < \begin{cases} \frac{11}{16}\delta^3 & \text{if } \delta \equiv 0 \pmod 4, \\ \frac{11}{16}\delta^3 + \frac{3}{16}\delta + \frac{1}{8} & \text{if } \delta \equiv 1 \pmod 4, \\ \frac{11}{16}\delta^3 + \frac{1}{4}\delta & \text{if } \delta \equiv 2 \pmod 4, \\ \frac{11}{16}\delta^3 + \frac{3}{16}\delta - \frac{1}{8} & \text{if } \delta \equiv 3 \pmod 4. \end{cases}$$

8 Sharper than complete bipartite

In this section, we prove that complete graphs are not exactly extremal for Problem 3. The complete bipartite graph $K_{\delta^k-1,k}$ is (k,δ)-choosable, but there are bipartite graphs with $\Delta_A = k$ and Δ_B smaller than $\delta^k - 1$ which are not (k,δ)-choosable.

Proposition 19. *For any $\delta, k \ge 2$, there is a bipartite graph $G = (V = A \cup B, E)$ with parts A and B having maximum degrees k and $f(\delta,k) < \delta^k$, respectively, that is not (k,δ)-choosable. Moreover, $f(\delta,k) \le \sum_{i=1}^{\delta} i^{k-1}$.*

Proof. We recursively construct bipartite graphs $G_i = (A_i \cup B_i, E_i)$ with parts A_i and B_i having maximum degree k and $\sum_{j=1}^{i}(\delta - j + 1)^{k-1}$. We simultaneously define a (k,δ)-list-assignment L_i of G_i such that there is some vertex $b_i \in B_i$ which can only be given one of $\delta - i$ colours out of its list in any proper L_i-colouring.

Let G_1 be the complete bipartite graph $K_{\delta^{k-1},k}$, and write $B_1 = \{v_1, \ldots, v_k\}$. For the vertices of B_1, we assign k disjoint lists of length δ, specifically, $L(v_j) = \{(j-1)\delta+1, \ldots, j\delta\}$ for $j \in [k]$. For the vertices of A_1, we assign as lists all possible k-tuples drawn from $\{1\}$ and $L(v_j)$ for $2 \le j \le k$. Since $b_1 := v_1$ cannot be given the colour 1 in any proper L_1-colouring, the conditions are satisfied for $i = 1$.

For the recursion, assume $i \ge 1$ and take the disjoint union of k copies of G_i, relabelling their (k, δ)-list-assignments so that their colour palettes are mutually disjoint. So the parts A_{i+1}, B_{i+1} of the bipartition so far include the disjoint unions of the k respective parts. Let $v_1, \ldots, v_k$ be the k copies of b_i, and for each $j \in [k]$ write $L'(v_j)$ for the set of $\delta - i$ colours to which the colour of v_j is restricted by assumption. By relabelling, we may assume $1 \in L'(v_1)$. We now add $(\delta - i)^{k-1}$ new vertices to A_{i+1} that are adjacent to every v_j. For these new vertices, we assign as lists all possible k-tuples drawn from $\{1\}$ and $L'(v_j)$ for $2 \le j \le k$. This completes the definition of G_{i+1} and L_{i+1}. By induction, $b_{i+1} := v_1$ may only be given a colour from $L'(v_1) \setminus \{1\}$ in any proper L_{i+1}-colouring, and moreover b_{i+1} is of maximum degree in B_{i+1}. So $(G_{i+1}, L_{i+1}, b_{i+1})$ satisfies the required conditions. This completes the recursive step.

The graph $G := G_\delta$ with parts $A = A_\delta$ and $B = B_\delta$ is not (k, δ)-choosable, since by construction we may not give any colour to b_δ in any proper L_δ-colouring. Furthermore, the maximum degrees in A and B are respectively k and $\sum_{i=1}^{\delta}(\delta - i + 1)^{k-1} = \sum_{i=1}^{\delta} i^{k-1}$, as required. □

9 Degrees and (k_A, k_B)-choosability

In this section, we give a condition on the minimum degree to conclude that a bipartite graph is not (k_A, k_B)-choosable, depending on the behaviour for complete bipartite graphs.

Theorem 20. *Suppose the complete bipartite graph $G_0 = (V{=}A_0 \cup B_0, E)$ with $|A_0| = a$ and $|B_0| = b$ is not (k_A, k_B)-choosable. Then any bipartite graph $G = (V{=}A \cup B, E)$ with parts A and B such that $|A| \le |B|$ and B has minimum degree $\delta_B > 4ab \log 4a \log k_A$ is not (k_A, k_B)-choosable.*

Proof. Let $\mathcal{F}_a$ and $\mathcal{F}_b$ be the collections of lists that can be assigned to A_0 and B_0, respectively, that certify non-(k_A, k_B)-choosability of G_0. Let $p = 1/(4b \log k_A)$. Randomly choose $X \subset A$, each vertex included independently with probability p. Then $\mathbb{E}(|X|) = p|A|$ and so by Markov's inequality,

$$\mathbb{P}(|X| > 2p|A|) < \frac{1}{2}.$$

Define a list-assignment L_X of X, by assigning to every vertex of X uniformly and independently a list of $\mathcal{F}_a$. Call a vertex v in B good if every member of $\mathcal{F}_a$ appears as a list on a neighbour (in X) of v. Note that

$$\frac{\mathbb{P}(\text{some } F \in \mathcal{F}_a \text{ does not appear for } v)}{a} \le \left(1 - \frac{p}{a}\right)^{\delta_B} \le \exp\left(-\frac{p}{a}\delta_B\right) < \frac{1}{4a}$$

implying that

$$\mathbb{P}(v \text{ is not good}) < \frac{1}{4}.$$

So by Markov's inequality,

$$\mathbb{P}\left(|\{v \mid v \text{ is not good}\}| > |B|/2\right) \leq \frac{\mathbb{E}(|\{v \mid v \text{ is not good}\}|)}{|B|/2} < \frac{1}{2}.$$

By the probabilistic method, there is some $X \subset A$ and a list-assignment L_X of X such that $|X| \leq 2p|A|$ and there are at least $|B|/2$ good vertices. Fix this choice and let B^* be the set of good vertices.

Fix an arbitrary L_X-colouring c_X of X. There are at most $k_A^{|X|}$ possibilities for the colouring c_X. Define a list-assignment L_{B^*} of B^*, by assigning to every vertex of B^* uniformly and independently a list of $\mathcal{F}_b$. Since every $v \in B^*$ is good, all lists of $\mathcal{F}_a$ appear in the neighbourhood of v and at least one choice of a list in $\mathcal{F}_b$ would imply that v cannot be properly coloured with a colour of that list. Hence the probability that every $v \in B^*$ can be properly L_{B^*}-coloured in agreement with c_X is at most

$$\left(1 - \frac{1}{b}\right)^{|B^*|} < \exp\left(-\frac{|B|}{2b}\right).$$

The probability that some proper colouring of G can completed given any L_X-colouring c_X is smaller than

$$k_A^{|X|} \exp\left(-\frac{|B|}{2b}\right) \leq \exp\left(2p|A| \log k_A - \frac{|B|}{2b}\right) \leq 1.$$

Thus by the probabilistic method there exists a list-assignment L_{B^*} of B^* such that no proper L_{B^*}-colouring can be found in agreement with any L_X-colouring. □

10 Conclusion

We have begun the investigation of an asymmetric form of list colouring for bipartite graphs. In one direction, we have found good general sufficient conditions through connections to independent transversals and to the coupon collector problem. This has incidentally yielded a non-trivial advance towards a difficult conjecture of Krivelevich and the first author. In another direction, we have established broad necessary conditions through an unexpected link between the bipartite choosability of complete graphs and a classic extremal set theoretic or design theoretic parameter. This link has fed naturally into the formulation of three attractive conjectures along these lines. Because of the rich connections this problem has to other important areas of combinatorial mathematics, we are hopeful that further study will lead to novel insights. We remark that Conjecture 7 comprises three asymptotic parameterisations of Problem 3 that we found most natural and interesting, all derived essentially from Theorem 6. There could be several other nice choices. Because the terrain is new, there are many interesting angles we have not yet had the opportunity to fully explore.

One possibility, based on the connection to combinatorial design theory, comes to mind. We have that $\overline{M}(2, q^2, q^2+q+1) = q^2+q+1$ for every prime power q due to the finite projective planes. With a small modification of the substitution of this fact into Corollary 12, we obtain that the complete bipartite graph $K_{\frac{1}{6}q^3(q+1)(q^2+q+1), q^2+q+1}$ is not $(3, q^2)$-choosable. On the other hand, Lemma 14 shows this is not that far from optimal, and in particular (2) shows that $K_{q^6/(80 \log q)^2, q^2+q+1}$ is $(3, q^2)$-choosable. It would be interesting to narrow the gap. For the specific case $q = 2$, a quick computer search checks that $K_{20,7}$ is not $(3, 4)$-choosable, but finding the largest r such that $K_{r,7}$ is $(3, 4)$-choosable seems difficult.

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A Extremal analysis of approximate Steiner systems

Proof of Theorem 13. First we prove the lower bound. Fix a family $\mathcal{F}$ of k_2-element subsets of $[\ell]$ with cardinality less than the leftmost expression. Choose C a k_1-element subset of $[\ell]$ uniformly at random. For any fixed $F \in \mathcal{F}$, we have

$$\mathbb{P}(F \cap C = \emptyset) = \binom{\ell - k_2}{k_1} \Big/ \binom{\ell}{k_1} = \frac{(\ell - k_2)!(\ell - k_1)!}{\ell!(\ell - k_1 - k_2)!}.$$

By a union bound and the choice of cardinality of $\mathcal{F}$,

$$\mathbb{P}(F \cap C = \emptyset \text{ for some } F \in \mathcal{F}) \le \sum_{F \in \mathcal{F}} \mathbb{P}(F \cap C = \emptyset) < 1.$$

So with positive probability there is a set C certifying that $\mathcal{F}$ has Property $\mathrm{A}(k_1, k_2, \ell)$.

Next we prove the upper bound. Fix C a k_1-element subset of $[\ell]$. Let F be a k_2-element subset of $[\ell]$ chosen uniformly at random. Then

$$\mathbb{P}(F \cap C = \emptyset) = \binom{\ell - k_1}{k_2} \Big/ \binom{\ell}{k_2} = \frac{(\ell - k_2)!(\ell - k_1)!}{\ell!(\ell - k_1 - k_2)!}.$$

Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a family of m k_2-element subsets of $[\ell]$ chosen uniformly at random. Based on the above calculation,

$$\mathbb{P}(F_i \cap C \neq \emptyset \text{ for all } i \in \{1, \dots, m\}) \leq \left(1 - \frac{(\ell - k_2)!(\ell - k_1)!}{\ell!(\ell - k_1 - k_2)!}\right)^m.$$

There are $\binom{\ell}{k_1}$ choices for C, so we have

$$\begin{aligned}&\mathbb{P}(\mathcal{F} \text{ contains a } C \text{ certifying Property A}(k_1, k_2, \ell))\\ &\leq \binom{\ell}{k_1} \exp\left(-m\frac{(\ell - k_2)!(\ell - k_1)!}{\ell!(\ell - k_1 - k_2)!}\right).\end{aligned}$$

This last expression is less than 1 if

$$m > \frac{\ell!(\ell - k_1 - k_2)!}{(\ell - k_2)!(\ell - k_1)!} \log \binom{\ell}{k_1},$$

which establishes the upper bound. □