Graph Powers

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Abstract

The investigation of the asymptotic behaviour of various parameters of powers of a fixed graph leads to many fascinating problems, some of which are motivated by questions in information theory, communication complexity, geometry and Ramsey theory. In this survey we discuss these problems and describe the techniques used in their study which combine combinatorial, geometric, probabilistic and linear-algebra tools.

1 Graph Powers

There are several known distinct ways to define the powers of a fixed graph. The n-th *AND power* of an undirected graph $G = (V, E)$ is the graph denoted by $G^{\wedge n}$ whose vertex set is V^n in which distinct vertices $(x_1 \dots x_n)$ and $(x'_1 \dots x'_n)$ are connected if $\{x_i, x'_i\} \in E$ for all $i \in \{1, 2, \dots, n\}$ such that $x_i \neq x'_i$. The n-th *OR power* of G is the graph denoted $G^{\vee n}$ whose vertex set is V^n in which distinct vertices $(x_1 \dots x_n)$ and $(x'_1 \dots x'_n)$ are connected if distinct x_i and x'_i are connected in G for some $i \in \{1, 2, \dots, n\}$.

The study of the asymptotic behaviour of various parameters of these powers of a fixed graph G, as well as their behaviour for similarly defined powers of directed and undirected graphs, is motivated by questions in various areas and leads to many intriguing problems. These are discussed in the following sections, in which we focus our attention mainly to the open problems in the area, and only briefly describe the known results and proof techniques. Proven and disproven conjectures are intermingled throughout the paper with open problems. More detailed proofs can be found in the papers listed in the bibliography.

2 Shannon Capacity

The *independence number* $\alpha(G)$ of a graph G is the maximum cardinality of a set of vertices of G no two of which are adjacent.

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A *channel* consists of a finite *input set* X, a (possibly infinite) *output set* Y, and a nonempty *fan-out set* $S_x \subseteq Y$ for every $x \in X$. In each channel *use*, a *sender* transmits an *input* $x \in X$ and a *receiver* receives an arbitrary *output* in S_x. Shannon [57] initiated the study of the amount of information a channel can communicate without error.

Associated with a channel C is its *characteristic graph* $G(C)$. Its vertex set is X and two (distinct) vertices are connected if their fan-out sets intersect, namely, both can result in the same output.

Note that every graph (V, E) is the characteristic graph of some channel: its input set is V, its output set is E, and S_v consists of all edges containing v.

The largest number of inputs a channel C can communicate without error in a single use is $\alpha(G(C))$, the independence number of its characteristic graph. This is done as follows: the sender and the receiver agree in advance on an independent set I. The sender transmits only inputs in I. Every received output belongs to the fan-out set of exactly one input in I, hence the receiver can correctly determine the transmitted input. Conversely, it is easy to see that a set containing two connected vertices cannot be communicated without error.

When the channel C is used $n > 1$ times the sender transmits a sequence $x_1 \ldots x_n$ of inputs and the receiver receives a sequence $y_1 \ldots y_n$ of outputs where each $y_i \in S_{x_i}$. Conceptually, n uses of C can be viewed as a single use of a larger channel C^n. Its input set is X^n, its output set is Y^n, and the fan-out set of $(x_1 \ldots x_n) \in X^n$ is the Cartesian product $S_{x_1} \times \ldots \times S_{x_n}$.

It is not difficult to check that the characteristic graph of C^n is simply $G^{\wedge n}$, the n-th AND product of $G = G(C)$.

It follows that the largest number of messages C can communicate without error in n uses is $\alpha(G^{\wedge n})$. The limit $\lim_{n \mapsto \infty}(\alpha(G^{\wedge n}))^{1/n}$ is called the *Shannon capacity of* G and is denoted by $c(G)$. It represents the number of distinct messages per use the channel can convey without errors, when used a large number of times. This limit exists, by super-multiplicativity, and is equal to the supremum $sup_{n \mapsto \infty}(\alpha(G^{\wedge n}))^{1/n}$. (It is worth noting that sometimes it is customary to call $\log c(G)$ the Shannon capacity of the graph G, but we prefer the definition above, following Lovász [50].)

By super-multiplicativity, $c(G) \geq \alpha(G)$ for every G. One of the early discoveries of Shannon was that sometimes strict inequality holds. Thus, for example, it is known by the results in [57] and [50] that the Shannon capacity of a cycle of length 5 satisfies $c(C_5) = \sqrt{5}$, whereas, of course, its independence number is 2. It seems interesting to decide how large can the gap between $\alpha(G)$ and $c(G)$ be. In particular, the following conjecture seems plausible.

Conjecture 2.1 *For every constant $C > 0$ there is a graph G with independence number 2 satisfying $c(G) > C$.*

As shown implicitly in [21] (and explicitly in [8]) there is a tight connection between the Shannon capacity of graphs and appropriate Ramsey numbers. The Ramsey number $r(3 : k)$ is the maximum number r such that there exists an edge coloring of the complete graph on r vertices by k colors with no monochromatic triangle. It turns out that the conjecture above is equivalent to the conjecture that for every $C > 0$ the Ramsey number $r(3 : k)$ exceeds C^k provided k is sufficiently large as a function of C. This is an old question of Erdős (c.f., e.g., [35], [19]). The best known lower bound for $r(3 : k)$ is $\Omega(321^{k/5})$, proved in [22] improving [29], and the best known upper bound is $O(k!)$ ([33]).

A related interesting question, which is more natural from the information theoretic point of view, is the estimation of the largest possible gap between $\alpha(G)$ and $c(G)$ as a function of the number n of vertices of the graph.

Conjecture 2.2 *For every $\epsilon > 0$ and every $n > n_0(\epsilon)$ there exists a graph G on n vertices satisfying $\alpha(G) < n^\epsilon$ and $c(G) > n^{1-\epsilon}$.*

Some results about this conjecture appear in [8], where it is shown that there are graphs G_n on n vertices satisfying $\alpha(G_n) \leq O(\log n)$ and $c(G_n) \geq \sqrt{n}$, but the assertion of the conjecture remains wide open and deserves further study.

Another intriguing problem is the computational problem of determining the Shannon capacity $c(G)$ of a given input graph G. It is not even known if this problem is decidable, and the value of $c(G)$ is not known even for several extremely simple graphs like the cycle of length 7 (or any odd cycle of length bigger than 5). It is also not known if the capacity of the complement of any odd cycle of length bigger than 5 is 2. The existence of a natural graph invariant which we cannot compute even for such simple finite graphs is highly intriguing, and although the results of Lovász [50], Haemers [38] and those in [3] supply some geometric and algebraic tools for studying this invariant, these methods work only in very limited cases, and it would be desirable to have additional effective techniques.

The following conjecture was raised by Shannon in his original paper in 1956.

Conjecture 2.3 ([57]) *The Shannon capacity of the disjoint union of two graphs is equal to the sum of their capacities.*

He proved that the capacity of a union of two graphs is always at least as large as the sum of the capacities, and equality holds provided the independence number of one of the graphs is equal to the chromatic number of its complement.

In [3] it is proved that this conjecture is false. There are two graphs, each having Shannon capacity at most k, so that the capacity of their disjoint union is at least $k^{\Omega(\log k/\log\log k)}$. The proof combines a variant of the beautiful construction of Frankl and Wilson [28], with the ideas in [4] and [8]. These counterexamples have an interesting and somewhat counterintuitive information theoretic interpretation. Indeed, If G and H are graphs of two channels, then their union represents the *sum* of the channels corresponding to the situation where either one of the two channels may be used, a new choice being made for each transmitted letter. Therefore such examples show that by using two channels alternately it is sometimes possible to obtain capacity that exceeds considerably the capacities of each of them. The problem of estimating the largest possible gap between the capacity of a union and the sum of the capacities remains open and deserves further study. A related problem is that of estimating the minimum possible value of $c(G) + c(\overline{G})$, where the minimum is taken over all graphs G with n vertices. The construction in [3] shows that this minimum is at most $e^{O(\sqrt{\log n \log\log n})}$, but it seems plausible that it is, in fact, $\Theta(\log n)$.

The relevance of geometry to the study of the Shannon capacity of graphs was initiated by Lovász in [50]. In this paper he introduced the θ-function of a graph $G = (V, E)$. One of its various equivalent definitions is the maximum possible value of the sum $\sum_{v\in V}(u_v^T \cdot b)^2$, where the maximum is taken over all unit vectors b and $\{u_v, v \in V\}$ in an arbitrary Euclidean space, so that for any two adjacent vertices w and v of G the vectors u_w and u_v are orthogonal. This function has many fascinating properties, including the fact that it is at least the independence number (and even at least the Shannon capacity) of G and at most the chromatic number of its complement, and the fact that it can be computed in polynomial time. See the survey [43] for more details.

A related problem to that of determining the maximum possible value of the Shannon capacity of a graph (on n vertices) with independence number 2 is that of determining or estimating the maximum possible θ-function of such a graph. This problem was raised by Lovász, and is equivalent to the geometric problem of estimating the maximum possible Euclidean norm of a sum of n unit vectors in R^n, so that among any three of them some two are orthogonal. Konyagin [45], and Kashin and Konyagin [44] showed that this maximum is at least $\Omega(n^{2/3}/(\log n)^{1/2})$ and at most $O(n^{2/3})$ and in [1] it is shown that it is $\Theta(n^{2/3})$. The corresponding graph of [1] has interesting Ramsey theoretic properties as well, and in particular it provides an explicit example of a triangle-free graph on n vertices with independence number $O(n^{2/3})$. Another example of a graph with this property has recently been constructed in [20].

Let $G(n, 1/2)$ denote, as usual, the random graph on n labeled vertices denoted $1, 2, \ldots, n$, obtained by picking each pair of distinct vertices i, j to be an edge randomly and independently with probability 1/2. We say that $G(n, 1/2)$ satisfies a property A *almost surely* if the probability that it satisfy A tends to 1 as n tends to infinity.

Conjecture 2.4 *There is an absolute constant b so that the Shannon capacity of $G(n, 1/2)$ is at most $b \log_2 n$ almost surely.*

As proved by Juhasz [41], the θ-function of $G(n, 1/2)$ is $\Theta(\sqrt{n})$ almost surely. This, together with the well known fact (cf., e.g., [16] or [9]), that the independence number of $G(n, 1/2)$ is $(2 + o(1)) \log_2 n$ almost surely, implies that the Shannon capacity of $G(n, 1/2)$ is at least $\Omega(\log n)$ and at most $O(\sqrt{n})$ almost surely.

3 Witsenhausen's Rate

The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors needed to color its vertices so that no two adjacent vertices have the same color.

A *dual source* consists of a finite set X, a (possibly infinite) set Y, and a *support set* $S \subseteq X \times Y$. In each *dual-source instance*, a *sender* P_X is given an $x \in X$ and a *receiver* P_Y is given a $y \in Y$ such that $(x, y) \in S$. Witsenhausen [61] and Ferguson and Bailey [25] initiated the study of the number of bits that P_X must transmit in the worst case in order for P_Y to learn x without error. (See [56] for the case when P_X and P_Y are allowed to interact.)

The *fan-out* of $x \in X$ is the set $S_x = \{y : (x, y) \in S\}$ of y's that are *jointly possible* with x. Associated with a dual source S is its *characteristic graph* $G = G(S)$. Its vertex set is X, and two (distinct) vertices x, x' are connected if their fan-out sets intersect, namely, there is a y that is jointly possible with both. Note that as is the case with the characteristic graphs of channels, here, too, every graph (V, E) is the characteristic graph of some dual source: $X = V$, $Y = E$, and $S = \{(x, y) : x \in y\}$.

The smallest number of possible messages that enable P_X to transmit one of them for a single instance of S is $\chi(G)$, the chromatic number of S's characteristic graph. To do so, P_X and P_Y agree in advance on a coloring of G. Given x, P_X transmits its color. P_Y, having y, can determine x because there is exactly one element of X with this color that is jointly possible with y. Conversely, it is easy to see that if two connected vertices are assigned the same message, an error can result.

In $n > 1$ instances of the dual source S, P_X knows $x_1 \dots x_n$ while P_Y knows $y_1 \dots y_n$ such that each $(x_i, y_i) \in S$ and wants to learn $x_1 \dots x_n$. Conceptually, n instances can be viewed as a single instance of a larger dual source whose support set is $S^n \subseteq X^n \times Y^n$. The characteristic graph of this larger dual source is $G^{\wedge n}$, the n-th normal power of the characteristic graph of S. It follows that the smallest number of messages that enable P_X to to convey n instances of S without error is $\chi(G^{\wedge n})$. Let $R(G)$ denote the limit $\lim_{n \mapsto \infty} (\chi(G^{\wedge n}))^{1/n}$. This quantity, called the *Witsenhaussen's rate* of G, measures the number of messages per instance that enable P_X to convey P_Y a large number of instances with no error. By sub-multiplicativity this limit exists and is always at most $\chi(G)$. (Here, too, it is sometimes customary to call $\log R(G)$ the rate of G but we prefer to avoid the logarithm). Witsenhausen [61] showed that for some dual sources, fewer messages suffice for each use of the larger system (than for a single instance). For example, for the cycle of length 5, $\chi(C_5) = 3 > \sqrt{5} = R(C_5)$.

It is interesting to study the maximum possible gap between $\chi(G)$ and $R(G)$. A result of Linial and Vazirani [48] implies that for every constant C there are graphs G for which $R(G) \leq 6$ while $\chi(G) > C$. In [8] it is shown that for every C and every $\epsilon > 0$ there are graphs G with $R(G) \leq 2 + \epsilon$ and $\chi(G) > C$. This is done by observing that the chromatic number of $G^{\wedge n}$ does not exceed that of $G^{\vee n}$, and by applying to the Kneser graphs the results of McEliece and Posner [52], and of Berge and Simonovits [13] that relate the chromatic number of $G^{\vee n}$ with the fractional chromatic number of G. This, together with the result of Lovász [49] about the chromatic number of Kneser graphs, gives the above mentioned bound.

The study of the maximum possible gap between $\chi(G)$ and $R(G)$ as a function of the number of vertices n of G seems more complicated. As shown in [8] the ratio between these two may be at least $\Omega(\sqrt{n}/\log^2 n)$, but the following conjecture remains open.

Conjecture 3.1 *For every $\epsilon > 0$ and $n > n_0(\epsilon)$ there is a graph G on n vertices satisfying $\chi(G) > n^{1-\epsilon}$ and $R(G) \leq n^{\epsilon}$.*

As is the case with the Shannon capacity, the computational problem of determining $R(G)$ for a given input graph G seems, in general, far beyond reach of the existing techniques, and even the computation of $R(G)$ for some simple, small graphs appears to be extremely difficult.

4 Cayley graphs

The partial results obtained in [8] concerning the study of Conjectures 2.2 and 3.1 combine algebraic and probabilistic techniques. The idea is to prove the existence of self complementary *Cayley graphs* in which the largest independent set is very small compared to the size of the graphs. For a group H and a subset $S \subset H$ satisfying $S = S^{-1}$, the *Cayley graph* of H with respect to S is the graph whose vertices are all elements of H and $g, h \in H$ are adjacent iff $h^{-1}g \in S$. The following Ramsey theoretic conjecture seems plausible.

Conjecture 4.1 *There exists an absolute constant b such that for every group H on n elements there is a Cayley graph of this group containing neither a complete subgraph nor an independent set on more than $b \log n$ vertices.*

A weaker version of this conjecture, obtained by replacing the $\log n$ term by a $\log^2 n$ term, is proved in [8], and this is the main tool in obtaining a partial result in the study of Conjecture 3.1. Although the solution of the last conjecture will not lead immediately to progress in the attempts to solve Conjecture 3.1, it is interesting in its own right.

Cayley graphs appear naturally in the study of the maximum possible value of the θ-function of a graph on n vertices with independence number 2. The construction in [1], in which this problem is solved up to a constant factor, uses Cayley graphs of Abelian groups. Their spectral properties, obtained via known bounds on character sums, play a crucial role in proving their properties. These questions are strongly related to problems in Combinatorial Geometry. In fact, by using related techniques together with some of the ideas in [24] and [14] it is proved in [10] that that there is an absolute positive constant $\delta > 0$, so that for all positive integers k and d, there are sets of at least $d^{\delta \log_2(k+2)/\log_2\log_2(k+2)}$ nonzero vectors in R^d, in which any $k+1$ members contain an orthogonal pair. This settles a problem of Füredi and Stanley [30]. The problem of estimating more accurately the maximum possible size of such a collection of vectors remains open.

5 Sperner Capacity

For a digraph $D = (V, E)$ and for a positive integer n, let $w(D^n)$ denote the maximum possible cardinality of a subset S of V^n in which for every ordered pair $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ of members of S there is some i, $1 \leq i \leq n$ such that (u_i, v_i) is a directed edge of D. It is easy to see that the function $g(n) = w(D^n)$ is super-multiplicative, and hence the limit

$$\mathop{lim}_{n \mapsto \infty} [\ (w(D^n))^{1/n}]$$

exists and is equal to the supremum of the quantity in the square brackets. This limit, denoted by $C(D)$, is called the *capacity* of the digraph.

The study of the capacity of directed graphs was introduced by Körner and Simonyi and by Gargano, Körner and Vaccaro in [47], [31], where the authors study the quantity $\sum(D) = \log C(D)$, which they call the *Sperner capacity* of D, and show that it generalizes the Shannon capacity of an undirected graph. In several subsequent papers [32], [46] they apply some properties of this invariant in the asymptotic solution of various problems in extremal set theory.

The problem of computing $C(D)$ for a given directed graph D is even more difficult in general than that of computing the Shannon capacity $c(G)$ of an undirected graph G , since if D is obtained from the complement of G by directing each edge in both directions then $C(D) = c(G)$. It is thus natural to try and restrict attention to some limited yet interesting classes of directed graphs. One such class is the class of all *tournaments*.

A *tournament* T is a digraph in which for every pair u, v of distinct vertices exactly one of the ordered pairs (u, v), (v, u) is a directed edge. The tournament T is *transitive* if there is a linear order on its vertices such that (u, v) is a directed edge iff u is smaller than v in this order.

It is easy to see that the capacity $C(T_n)$ of the transitive tournament on n vertices is n. Therefore, the capacity of any tournament that contains a transitive subtournament on n vertices is at least n. Using algebraic techniques, Calderbank, Frankl, Graham, Li and Shepp [18] proved that the capacity of the cyclically directed triangle is 2, namely, the number of vertices in the largest transitive subtournament in it. Blokhuis [15] gave a simpler proof of this result. This inspired Körner and Simonyi ([46]) to conjecture that for every tournament T, the capacity $C(T)$ is the maximum number of vertices in a transitive subtournament of T.

Using probabilistic arguments it is shown in [2] that this conjecture is false. This is based on some simple properties of *random tournaments* obtained by deciding, for each pair of vertices u,v randomly and independently, if (u, v) or (v, u) is a directed edge with equal probability. However, the assertion of the conjecture is true for all tournaments with at most 5 vertices. In order to prove the assertion of the conjecture for small tournaments one can develop an algebraic method, related to the technique of Haemers in [38], which enables one to bound the capacities of other digraphs as well.

The following conjecture seems plausible.

Conjecture 5.1 *There is an absolute constant c so that the probability that the capacity of a random tournament on n vertices exceeds $c \log_2 n$ tends to 0 as n tends to infinity.*

Let $t(T)$ denote the maximum number of vertices in a transitive subtournament of T. It would be interesting to estimate the maximum possible value of the ratio $C(T)/t(T)$, as T ranges over all tournaments of n vertices. It can be shown that this maximum is at least $\Omega(\sqrt{n}/\log n)$, but we suspect it might be bigger.

Another interesting problem, suggested by Körner, is to characterize all tournaments T in which for every subtournament T', $C(T') = t(T')$.

Given a directed graph $D = (V, E)$ and a field F, a *representation of D of dimension d over F* is an assignment of a vector $u_v \in F^d$ to each vertex v of D, so that no vector u_v lies in the span of the vectors assigned to the in-neighbors of v. This notion seems promising in the study of the capacities $C(D)$ of directed graphs, as it can be shown, using some linear-algebra tools, that if D has such a representation (over any field) then its capacity $C(D)$ is at most d. This result, obtained jointly by the author and G. Tardos, together with some simple combinatorial arguments, suffices to determine the capacity of all tournaments with at most 5 vertices. It can be used to show that, as proved in [2], the Sperner capacity of any digraph with maximum outdegree d is at most $d+1$. More generally, it implies that, as proved in [27], the Sperner capacity of any digraph whose vertex set can be partitioned into k classes, where d_i is the maximum outdegree in the i-th class, is at most $\sum_{i=1}^{k}(d_i + 1)$. It does not suffice, however, to solve conjecture 5.1, whose study seems to require additional techniques. Indeed, the minimum possible value of $\sum_{i=1}^{k}(d_i+1)$, where the minimum is taken over all tournaments on n vertices and all partitions of their vertices, where the numbers d_i are defined as above, is easily seen to be at least $(n+1)/2$. Combining probabilistic and combinatorial arguments one can show that the probability that the capacity of a random tournament on n vertices is bigger than ϵn tends to 0 for every fixed positive ϵ, as n tends to infinity, but the assertion of the conjecture is far stronger.

Sali and Simonyi [58] proved that any undirected vertex transitive self-complementary graph on n vertices has an orientation whose Sperner capacity is $\sqrt{n}$. As mentioned in their paper, it might be that any undirected graph G has an orientation D whose Sperner capacity is equal to the Shannon capacity of the complement of G.

6 Additional Graph Powers

The n-th *(sparse) power* of an undirected graph $G = (V, E)$, denoted simply by G^n, is the graph whose vertex set is V^n in which distinct vertices $(x_1 \ldots x_n)$ and $(x'_1 \ldots x'_n)$ are connected iff there exists a single index i such that $x_j = x'_j$ for all $j \neq i$ and x_i and x'_i are connected in G. The study of the asymptotic behaviour of the independence number of G^n, for a fixed graph G, can be motivated by the search for appropriately defined error-detecting codes in a certain channel. Indeed, if the vertices of the graph are the possible inputs, and each input may be altered, during the transmission process in a channel, to one of its neighbors, then this independence number represents the maximum number of messages of length n that can be sent in a way that enables us to detect the occurrence of one error. In [36] and some of its references the authors obtained several results about the problem of estimating these independence numbers. In particular, they proved that for every fixed graph G, the limit $\lim_{n \mapsto \infty} \frac{\alpha(G^n)}{|V|^n}$ exists, and is at least $1/\chi(G)$ and at most the reciprocal of the fractional chromatic number of G. Moreover, for every Cayley graph of an Abelian group this limit is precisely the reciprocal of the fractional chromatic number of G. Several examples suggest that this invariant has interesting properties and its study seems to require algebraic and combinatorial tools.

Another interesting topic dealing with sparse powers is the study of isoperimetric inequalities in such powers. For $G = (V, E)$ and G^n as above, and for an integer d, let $f(G, n, d)$ denote the maximum possible number of vertices in V^n of distance at least d from a set of half the vertices of V^n, divided by $|V|^n$. The study of the asymptotic behaviour of $f(G, n, d)$ attracted a considerable amount of attention, as it appears naturally in the study of many combinatorial and geometric problems. See, for example, [53, 7, 55, 54, 17, 59, 37]. In all these references it is shown that for every fixed G there exists a positive constant $b = b(G)$ such that $f(G, n, d) \leq \exp[-b(d^2/n)]$ for all d and n, but it seems interesting to get a more precise estimate, like the tight inequality of Harper [39] for powers of a single edge. This inequality asserts that for the graph of the n-cube G where n is, say, $2k+1$, $f(G, n, d)$ is precisely the set of all vertices of the cube whose distance from a Hamming ball of radius $(n-1)/2$ is at least d. That is, in this case

$$f(G, n, d) = \sum_{i=0}^{\frac{n+1}{2}-d} \binom{n}{i}.$$

In [5] the authors obtain, for every fixed G, an asymptotic formula *with the right constant* for $\log f(G, n, d)$ if n is large and d is not too small as a function of n. This is done by a combination of large deviation techniques and ideas borrowed from Game Theory, and provides an asymptotic formula for all relevant $d >> \sqrt{n}$.

The results for $d = o(n)$ are somewhat simpler and lead to a definition of a constant called in [5] the spread constant of a graph G, which appears in the expression for $f(G, n, d)$ for values of $d >> \sqrt{n}$ that satisfy $d = o(n)$. For a graph $G = (V, E)$, call a function X mapping V to the set of reals *Lipschitz* if $|X(u) - X(v)| \leq 1$ for all $uv \in E$. Any such function can be considered as a random variable on the symmetric probability space V (in which each vertex has the same probability) and as such has a variance $VAR[X]$. The *spread constant* $c = c(G)$ is the maximum possible value of $VAR[X]$ over all Lipschitz X. In [5] it is proved that for every fixed graph G, if $d/\sqrt{n}$ and n/d tend to infinity, then

$$f(G, n, d) = e^{-\frac{d^2}{2cn}(1+o(1))}.$$

For linear distances the situation is more complicated. Let $m = m(G)$ denote the maximum, over all vertices v of G, of the average distance of a vertex of G from v. It is not too difficult to check that if $d \geq mn + \sqrt{cn}$ then $f(G, n, d) = 0$. For values of d satisfying $\sqrt{n} << d$ and $mn - d >> \sqrt{n}$ the asymptotic value of $f(G, n, d)$ is given as follows. For a real λ, let $L(\lambda)$ be the maximum possible value of $\ln E[e^{\lambda X}]$, where the maximum is taken over all Lipschitz $X : V \mapsto \mathbf{R}$ with expectation $E[X] = 0$. For a real t define

$$R(t) = sup_{\lambda \in R}[\lambda t - L(\lambda)].$$

Then, for d as above,

$$f(G, n, d) = e^{-R(d/n)n(1+o(1))}.$$

If, for example, G is a triangle, then a simple though tedious computation shows that for every $0 \leq t < 2/3$,

$$e^{-R(t)} = 2(2 - 3t)^{t-2/3}(6t + 2)^{-1/3-t},$$

supplying a tight isoperimetric inequality for the space of all vectors of length n over an alphabet of 3 letters with the Hamming metric.

The *tensor product* of two graphs $G = (V, E)$ and $H = (V', E')$ is the graph whose vertex set is the Cartesian product $V \times V'$ of the vertex sets of G and H, where (u, u') is joined to (v, v') if either $uv \in E$ or $u'v' \in E'$, but not both. The *n-th tensor power of* G is the tensor product of n copies of G. These powers are studied in [60], where the author shows how to use their properties to construct edge colorings of complete graphs by two colors in which the number of monochromatic copies of K_4 is smaller than the expected number of such copies in random colorings. Similar results hold for other graphs including all graphs containing a K_4, as shown in [40]. This disproves conjectures of Erdös [23] and of Burr and Rosta [12].

7 Some computational aspects

One of the main reasons for the fast development of Combinatorics during the recent years is certainly the widely used application of combinatorial methods in the study and the development of efficient algorithms. It is therefore natural to study the computational aspects of the problems considered here.

The properties of the Lovász θ-function and the fact that it can be computed in polynomial time using semidefinite programming (see [34]) suggest that it can be used for the design of approximation algorithms for the chromatic number of a graph and for its independence number. It is well known that if, as is widely believed, the complexity classes P and NP differ, then there is no polynomial time algorithm that provides a reasonable approximation for these quantities (see [26], [11], [51]). In addition, examples of Feige [24] show that the ratio between the θ-function of a graph on n vertices and its independence number (or chromatic number) may be as large as $n/e^{O(\sqrt{\log n})}$. Despite these facts it is possible to apply the properties of the θ-function and obtain some meaningful approximation of these invariants for graphs with small chromatic number or with large independence number, and indeed there are algorithms along these lines for both problems developed in [42] for coloring and in [6] for the independence number. A better understanding of the possible gaps between the θ-function and these two parameters may lead to improved algorithms.

The computational problem of determining the Shannon capacity of a given input graph seems very difficult. It would be interesting, however, to develop any (possibly exponential or even doubly exponential time) algorithm that provides a reasonable approximation for this quantity . The techniques in [8] or [3] can be extended to show that the independence number of any fixed AND-power of a large graph provides essentially no information on its Shannon capacity. We conclude the paper with the following plausible conjecture

Conjecture 7.1 *There exists a function $f(n) \leq O(2^n)$ such that for any graph G on n vertices, $(\alpha(G^{\wedge f(n)}))^{1/f(n)}$ is always at least $0.99c(G)$.*

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