Hardness of Fully Dense Problems

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Abstract

In the past decade, there has been a stream of work in designing approximation schemes for dense instances of NP-Hard problems. These include the work of Arora, Karger and Karpinski from 1995 and that of Frieze and Kannan from 1996. We address the problem of proving hardness results for (fully) dense problems, which has been neglected despite the fruitful effort put in upper bounds. In this work we prove hardness results of dense instances of a broad family of CSP problems, as well as a broad family of ranking problems which we refer to as CSP-Rank. Our techniques involve a construction of a pseudorandom hypergraph coloring, which generalizes the well-known Paley graph, recently used by Alon to prove hardness of feedback arc-set in tournaments.

1 Introduction

Dense instances of MAX-SNP problems are known to be easier to approximate than the general case [4, 5, 8, 12–15, 17]. In 1995, Arora et al. [8] proved that there exist approximation schemes for MAX-SNP problems for dense instances, and introduced the technique of smooth programs. Later in 1996, Frieze et al. [14] proved an efficient verion of the regularity lemma and used it as a general framework for approximation scheme for MAX-SNP problems. (Some earlier special cases have been treated already in [4].) More recently, Ailon et al. [1] proved that finding the minimum feedback arc-set of a tournament (fully dense digraph) can be approximated to within a constant factor of 2.5, where the best known algorithm for general digraph was $O(\log n \log\log n)$ [11, 19].

To the best of our knowledge, there are only few hardness results for dense or fully dense instances. NP-Hardness of the minimum feedback arc set in tournaments (fully-dense digraphs) was conjectured by Bang-Jensen and Thomassen [9] (NP-Hardness of the general digraph case was well known [16]). A first step in proving the conjecture was taken by Ailon et al. [1], by demonstrating a poly-time randomized reduction from hard digraphs to tournaments. Thus, NP $\nsubseteq$ BPP if and only if there is no poly-time algorithm for minimum feedback arc-set in tournaments. Alon [3] derandomized this reduction, consequently proving Bang-Jensen and Thomassen's conjecture with no assumption. The randomized reduction and its derandomization can be informally explained as follows. Start with a hard digraph G, and blow it up by a factor of some integer a by creating a group of a copies of each vertex, and for any edge e in G connect the two groups corresponding to

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the vertices incident to e by a complete bipartite digraph (with the same orientation as e). This blow-up is not a tournament but it can be made a tournament by randomly and independently orienting all non-edges. The main idea is, that the rate of growth of the hardness (with respect to a) dominates the "noise" introduced by the random edges. The derandomization is done by choosing the orientation of non-edges according to the Paley tournament. The Paley tournament (see, for example, [7], Chapter 9) is an algebraically constructed tournament possessing pseudorandom properties that are required for the reduction.

In this work we address two types of problems, which we refer to as r-CSP$_{\mathcal{F}}$ and r-CSP-RANK$_{\mathcal{F}}$. The former is parameterized by an integer r and a finite family of real valued functions $\mathcal{F}$ on r boolean variables. An instance of the problem involves a set of clauses, each clause contains a subset of r variables in a set of n variables, and a function $f \in \mathcal{F}$. Given an assignment of boolean values to the variables, the value of the objective function is the total sum over all clauses of the evaluation of f on its corresponding variables. The goal is to minimize the objective function.

In r-CSP-RANK$_{\mathcal{F}}$, the functions $f \in \mathcal{F}$ are from the domain of permutations on r objects. Given a ranking (permutation) of the n variables, the value of the objective function is the total sum over all clauses of the evaluation of f on the induced ranking of its corresponding variables, and the goal is to minimize the objective function.

In fully dense instances of r-CSP$_{\mathcal{F}}$ and r-CSP-RANK$_{\mathcal{F}}$, which we refer to as r-CSP-FULL$_{\mathcal{F}}$ and r-CSP-RANK-FULL$_{\mathcal{F}}$, respectively, there is a clause corresponding to each possible choice of r variables. In Section 4 we show that under a certain assumption of homogeneity that will be explained later, if an r-CSP-RANK$_{\mathcal{F}}$ problem is NP-Hard, then its r-CSP-RANK-FULL$_{\mathcal{F}}$ version is NP-Hard to approximate to within an additive error of $n^{r-\varepsilon}$ for any $\varepsilon > 0$, where n is the number of variables.

Examples of NP-Hard homogenous r-CSP$_{\mathcal{F}}$'s are MAX-r-LIN-2 (maximizing the number of satisfied equations modulo 2, each equation involving exactly r variables), and MAX-r-CNF (maximizing the number of satisfied CNF clauses, each clause involving exactly r variables).

Examples of NP-Hard homogenous r-CSP-RANK$_{\mathcal{F}}$'s are FEEDBACK-ARC-SET and BETWEENNESS . In FEEDBACK-ARC-SET we are given a digraph, and the goal is to rank its vertices minimizing the number of *backward* edges with respect to the ordering. The fully dense version (on tournaments) is referred to as FEEDBACK-ARC-SET-TOUR. In BETWEENNESS we are given betweenness constraints on a ground set of n elements. Each constraints involves 3 elements and a designated element among the three. The objective function of a ranking of the elements is the number of betweenness constraints for which the designated element is not between the other two with respect to the ranking, and the goal is to minimze the objective function. We refer to the fully dense instance (i.e. a betweenness constraint for all choices of three elements) as BETWEENNESS-TOUR. The problem BETWEENNESS is known to be NP-Hard [18], and furthermore, it is known that ranking a betweenness instance in which all the constraints are consistent (i.e. there exists a ranking satisfying all of them) is NP-Hard [10][1].

The proofs of our main theorems demonstrate reductions from the general dense instances to the fully dense instances, using the basic approach of [1] and [3], and including several additional ideas that are needed to handle the more general situation. A sparse hard instance is blown up, and a fully dense instance is created as a hybrid of the blow-up and a pseudorandom structure that simulates a random choice of non-clauses in the original instance.

The paper is organized as follows. In Section 2 we construct an algebraic coloring of a complete r-regular hypergraph which will be the source of pseudorandomness for all structures that are considered in what follows. This coloring is a generalization of the Paley graph construction. In

[1]Note however that ranking a fully-dense consistent BETWEENNESS-TOUR instance is very easy

Section 3 we discuss r-CSP$_{\mathcal{F}}$, in Section 4 we discuss r-CSP-RANK$_{\mathcal{F}}$, and in Section 5 we discuss some open problems and remarks.

2 The Basic Construction

Definition 2.1 *Let $s > 1, r > 1$ be integer constants, and let p be a prime such that $s|p-1$. Let g denote a generator of Z_p^*. Let H denote the subgroup of Z_p^* generated by g^s. Denote by H_i the coset Hg^i for $i = 0, \ldots, s-1$. For an element $j \in Z_p^*$, define $[j]_p^s = i$ if $j \in H_i$, and formally $[0]_p^s = -\infty$. Let $\omega = e^{(2\pi i)/s}$ be a primitve s'th complex root of unity. For $0 \le t < s$ let*

$$\chi_t(j) = \begin{cases} \omega^{t \cdot [j]_p^s} & j \in Z_p^* \\ 0 & j = 0 \end{cases} .$$

The s-adic residual coloring of the complete r-regular hypergraph on Z_p is defined as follows.: The color of the hyperedge $\mathbf{j} = (\mathbf{j}_1, \ldots, \mathbf{j}_r)$ is

$$col_p^s(\mathbf{j}) = \max\{[\mathbf{j}_1 + \cdots + \mathbf{j}_r]_p^s, 0\} .$$

Note that this coloring allows considering the hyperedges as multisets (hyper-self-loops), but we will only consider r-regular hypergraphs with no hyper-self-loops in what follows.

We assume the following notation. We will use the linear order $<$ on $Z_p = \{0, \ldots, p-1\}$ induced by the order on integers. For a set B, we let $\binom{B}{r}$ denote the family of all subsets of size r of B. If $B \subseteq Z_p$, we identify $\binom{B}{r}$ with the set of vectors $\mathbf{i} = (\mathbf{i}_1, \ldots, \mathbf{i}_r) \in B^r$ such that $\mathbf{i}_1 < \cdots < \mathbf{i}_r$. For an integer m, let $S(m)$ denote the set of permutations on the integers $\{1, \ldots, m\}$.

The s-adic residue coloring behaves like a randomized coloring in the sense that for any big disjoint sets $A_1, \ldots, A_r \subseteq Z_p$, roughly $1/s$ fraction of the hyperedges in $A_1 \times \cdots \times A_r$ have color i for all $i = 0, \ldots, s-1$. We make this precise in what follows. First, we state some lemmas and definitions, starting with the following well known fact.

Lemma 2.2 *Fix distinct $j, l \in Z_p$, and $1 \le t < s$.*

$$\sum_{i \in Z_p} \chi_t(i+j)\overline{\chi_t(i+l)} = -1 ,$$

where $\bar{(\cdot)}$ denotes complex conjugation.

Proof:

$$\begin{aligned} \sum_{i \in Z_p} \chi_t(i+j)\overline{\chi_t(i+l)} &= \sum_{i \neq -l, -j} \chi_t(i+j)\overline{\chi_t(i+l)} \\ &= \sum_{i \neq -l, -j} \chi_t(\frac{i+j}{i+l}) \\ &= \sum_{i \neq -l, -j} \chi_t\left(1 + \frac{j-l}{i+l}\right) = -1. \end{aligned}$$

The last equality is because as i takes all values in $Z_p \setminus \{-j, -l\}$, the expression $1 + (j-l)/(i+l)$ takes all values in Z_p^* except 1, and it is well known that the sum of χ_t over all values in Z_p^* is 0. □

[7].

Assuming p, s, r fixed, for a set $A \subseteq \binom{Z_p}{r}$ let $n_j(A)$ denote the number of combinations $\mathbf{i} \in A$ such that $col_p^s(\mathbf{i}) = j$ for $j = 0, \ldots, s-1$. That is, if we define $m_j(A) = \left|\{\mathbf{i} \in A | [\mathbf{i}_1 + \cdots + \mathbf{i}_r]_p^s = j\}\right|$, then $n_j(A) = m_j(A)$ for $j > 0$ and $n_0(A) = m_0(A) + m_{-\infty}(A)$.

Lemma 2.3 *Let $A_1, \ldots, A_r$ be $r \geq 2$ subsets of Z_p, and let $A = A_1 \times \cdots \times A_r$. Then for all $1 \leq t < s$,*

$$\left|\sum_{\mathbf{i} \in A_1} \chi_t(\mathbf{i}_1 + \cdots + \mathbf{i}_r)\right| \leq |A|^{1/2} p^{(r-1)/2} \ .$$

Proof: Using Cauchy-Schwartz and Lemma 2.2 and letting I substitute $(\mathbf{i}_1 + \cdots + \mathbf{i}_{r-1})$ and $\sum_{(*)}$ substitute $\sum_{\mathbf{i} \in Z_p^{r-1}}$,

$$\begin{aligned}
&\left|\sum_{\mathbf{i} \in A} \chi_t(\mathbf{i}_1 + \cdots + \mathbf{i}_r)\ \right|^2 \\
&\leq |A_1| \cdots |A_{r-1}| \sum_{\mathbf{i}_1 \in A_1} \cdots \sum_{\mathbf{i}_{r-1} \in A_{r-1}} \left|\sum_{\mathbf{i}_r \in A_r} \chi_t(I + \mathbf{i}_r)\right|^2 \\
&\leq |A_1| \cdots |A_{r-1}| \sum_{(*)} \left|\sum_{\mathbf{i}_r \in A_r} \chi_t(I + \mathbf{i}_r)\right|^2 \\
&= |A_1| \cdots |A_{r-1}| \left(p^{r-1}|A_r| + 2\sum_{(*)} \sum_{j < l \in A_r} \chi_t(I + j)\overline{\chi_t(I + l)}\right) \\
&= |A| p^{r-1} + 2|A_1| \cdots |A_{r-1}| p^{r-2} \sum_{i \in Z_p} \sum_{j < l \in A_r} \chi_t(i + j)\overline{\chi_t(i + l)} \\
&\leq |A| p^{r-1} + |A_1| \cdots |A_{r-1}| p^{r-2} |A_r| (|A_r| - 1) \cdot (-1) \\
&= |A| \left(p^{r-1} - p^{r-2}(|A_r| - 1)\right) \\
&\leq |A| p^{r-1} \ ,
\end{aligned} \qquad (1)$$

as required. □

The assertion of the last lemma for $r = 2$ is proved, for example, in [2], and the result for bigger r can be deduced from this case by the triangle inequality. The short proof above is included in order to make the paper self contrained.

Lemma 2.4 *Let $A_1, \ldots, A_r, A$ be as in Lemma 2.3, and assume in addition that $A_1, \ldots, A_r$ are pairwise disjoint, and that $A_1 < A_2 \ldots < A_r$, that is, each element of A_j is smaller than each element of $A_{j'}$ for all $j < j'$. (Thus A can be viewed as a subset of $\binom{Z_p}{r}$.) Then for all $j = 0, \ldots, s-1$,*

$$|n_j(A) - |A|/s| \leq 2(|A|)^{1/2} p^{(r-1)/2}$$

Proof:

By Lemma 2.3, we have that for all $1 \leq t < s$,

$$\left|\sum_{j=0}^{s-1} m_j(A)\omega^{tj}\right|^2 \leq |A|p^{r-1} \ .$$

Therefore, for all $0 \leq t < s$,

$$\left|\sum_{j=0}^{s-1} (m_j(A) - |A|/s)\omega^{tj}\right|^2 \leq |A|p^{r-1} \ .$$

$$\sum_{t=0}^{s-1}\left|\sum_{j=0}^{s-1} (m_j(A) - |A|/s)\omega^{tj}\right|^2 \leq s|A|p^{r-1} \ .$$

By Parseval's Theorem,

$$\sum_{j=0}^{s-1} (m_j(A) - |A|/s)^2 \leq |A|p^{r-1} \ .$$

Therefore, $|(m_j(A) - |A|/s)| \leq |A|^{1/2}p^{(r-1)/2}$ for $j = 0, \ldots, s-1$. To complete the proof, we must show that $m_{-\infty}(A) \leq |A|^{1/2}p^{(r-1)/2}$. Clearly, $m_{-\infty}(A) \leq |A_1|\cdots|A_{r-1}| \leq |A|^{1/2}p^{(r-1)/2}$, as required. We conclude that $|(n_j - |A|/s)| \leq 2|A|^{1/2}p^{(r-1)/2}$ for all $j = 0, \ldots, s-1$.

Lemma 2.5 *Suppose $r \geq 2$ and let $B_1, \ldots, B_r \subseteq Z_p$ be some subsets. let*

$$A = A(B_1, \ldots, B_r) = \left\{ \mathbf{i} \in \binom{Z_p}{r} \middle| \mathbf{i}_j \in B_j, \ j = 1, \ldots, r \right\} \ .$$

Then

$$|n_j(A) - |A|/s| \leq c_r(|B_1|\cdots|B_r|)^{1/2}(\log|B_1|\cdots|B_r|)^{r-1}p^{(r-1)/2} \ , \tag{2}$$

for some global $c_r > 0$ that depends only on r.

Proof: Let k be such that $|B_k| \geq |B_j|$ for $j = 1, \ldots, r$.

We can assume that $|B_k| \geq 4$, because we can always find $c_r > 0$ that will make (2) true for the case $|B_j| < 4$ for $j = 1, \ldots, r$.

Sort the elements of B_k to obtain a sequence $b_0 < \cdots < b_{n-1}$, where $n = |B_k|$. Let x be an integer such that $\lceil |B_k|/2 \rceil$ elements of B_k are less than x, and $\lfloor |B_k|/2 \rfloor$ are at least x. For each $j = 1, \ldots, r$, let B_j^L denote $\{y \in B_j | y < x\}$ and $B_j^r = \{y \in B_j | y \geq x\}$.

We divide the set A to subsets $S \subseteq A$, such that each S is a product $S_1 \times \cdots \times S_r$ for some subsets $S_j \subseteq B_j$ such that $S_1 < S_2 < \cdots < S_r$. This will enable us to use Lemma 2.4. Our construction will be inductive over $|B_k|$ and r. More precisely, we construct a family $\mathcal{S}(B_1, \ldots, B_r)$ such that $S \subseteq A = A(B_1, \ldots, B_r)$ for any $S \in \mathcal{S}(B_1, \ldots, B_r)$, each such S is a product of r disjoint subsets as described above, any distinct $S_1, S_2 \in \mathcal{S}(B_1, \ldots, B_r)$ are disjoint and $\cup_{S \in \mathcal{S}(B_1,\ldots,B_r)} S = A$. First, for $r = 1$ we define $\mathcal{S}(B_1)$ as the trivial decomposition $\mathcal{S}(B_1) = \{A\}$. For $r > 1$, $|B_k| \leq 4$, we define $\mathcal{S}(B_1, \ldots, B_r)$ by the singleton decomposition $\{\{\mathbf{i}\} \mid \mathbf{i} \in A\}$, for all r. For $r > 1$, $|B_k| > 4$, we define

$$\mathcal{S}(B_1, \ldots, B_r) = \bigcup_{r'=0}^{r} \mathcal{S}(B_1^L, \ldots, B_{r'}^L) \times \mathcal{S}(B_{r'+1}^R, \ldots, B_r^R) \ ,$$

where $\mathcal{S}(B_1^L,\ldots,B_{r'}^L)\times\mathcal{S}(B_{r'+1}^R,\ldots,B_r^R)$ is the collection of all $S_0\times S_1$ such that $S_0\in\mathcal{S}(B_1^L,\ldots,B_{r'}^L)$ and $S_1\in\mathcal{S}(B_{r'+1}^R,\ldots,B_r^R)$. The desired properties of $\mathcal{S}(B_1,\ldots,B_r)$ can be easily verified using structural induction.

By the triangle inequality, we conclude that

$$|n_j(A)-|A|/s| \le \sum_{S\in\mathcal{S}(B_1,\ldots,B_r)} |n_j(S)-|S|/s| \ .$$

By Lemma 2.4 we get that for all $S\in\mathcal{S}(B_1,\ldots,B_r), r\ge 2$,

$$|n_j(S)-|S|/s| \le 2|S|^{1/2}p^{(r-1)/2} \ .$$

Therefore,

$$|n_j(A)-|A|/s| \le 2p^{(r-1)/2}\sum_{S\in\mathcal{S}(B_1,\ldots,B_r)} |S|^{1/2} \ .$$

So it suffices to show that

$$\begin{aligned} f(B_1,\ldots,B_r) &= \sum_{S\in\mathcal{S}(B_1,\ldots,B_r)} |S|^{1/2} \\ &\le c_r(|B_1|\cdots|B_r|)^{1/2}(\log|B_1|\cdots|B_r|)^{r-1} \ , \end{aligned} \tag{3}$$

for all $r\ge 2$. This can be proven using structural induction on $\mathcal{S}(B_1,\ldots,B_r)$, for $r\ge 1$, using the following relations which are immediate to verify. We omit the details of the induction.

1. If $|B_k|\le 4$ then $f(B_1,\ldots,B_r)\le c'_r(|B_1|\cdots|B_r|)^{r/2}(\log|B_1|\cdots|B_r|)^{r-1}$ for some $c'_r>0$ that depends only on r.

2. If $|B_k|>4$ then

$$\begin{aligned} f(B_1,\ldots,B_r) &= f(B_1^L,\ldots,B_r^L)+f(B_1^R,\ldots,B_r^R) \\ &\quad+\sum_{r'=1}^{r-1} f(B_1^L,\ldots,B_{r'}^L)f(B_{r'+1}^R,\ldots,B_r^R) \end{aligned}$$

Corollary 2.6 *In the notation of Lemma 2.5,*

$$|n_j(A)-|A|/s| \le c_r(|B_1|\cdots|B_r|)^{1/2}(r\log p)^{r-1}p^{(r-1)/2} \ . \tag{4}$$

Note: The bounds in Lemmas 2.4, 2.5 and Corollary 2.6 are not tight and can be improved, but they suffice for our purpose here.

3 r-CSP

An r-CSP$_{\mathcal{F}}$ instance consists of the following elements.

1. A finite family $\mathcal{F}$ of constraints $f:\{0,1\}^r\to\mathbb{R}$, where r is a constant.

2. An r-uniform hypergraph $H=(V,E,\beta)$ with vertex set V and edge set E, equipped with a function $\beta:E\to\mathcal{F}$. We will assume that $V\subseteq\mathbb{Z}$ in what follows.

We consider fixed $r, \mathcal{F}$ which are not part of the input. For an assignment $\alpha : V \to \{0,1\}$ of boolean values to V, we associate an objective function $\mathrm{FIT}(H, \alpha)$, defined as

$$\mathrm{FIT}(H, \alpha) = \sum_{\mathbf{i} \in E} \beta(\mathbf{i})(\alpha(\mathbf{i}_1), \ldots, \alpha(\mathbf{i}_r)) \ .$$

In the last sum, we assume that $\mathbf{i}_1 < \cdots < \mathbf{i}_r$ to allow a canonical passing of arguments to $f \in \mathcal{F}$. We say that $\mathcal{F}$ is *homogenous* if for any fixed $\mathbf{x} \in \{0,1\}^r$,

$$\sum_{f \in F} f(\mathbf{x}) = 0 \ .$$

We will assume in what follows that $\mathcal{F}$ is *always* homogenous.

Let r-CSP-FULL$_\mathcal{F}$ denote the problem r-CSP$_\mathcal{F}$ restricted to fully dense instances, that is, instances in which $E = \binom{V}{r}$.

Our main result is as follows.

Theorem 3.1 *If r-CSP$_\mathcal{F}$ is NP-Hard to approximate to within an additive error of 1 for some fixed $r, \mathcal{F}$, then r-CSP-FULL$_\mathcal{F}$ is NP-Hard to approximate to within an additive factor of $n^{r-\varepsilon}$ for any $\varepsilon > 0$, where n is the number of vertices in the hypergraph.*

To prove Theorem 3.1 we demonstrate a reduction that will make use of the residual coloring defined in Section 2. Let $s = |\mathcal{F}|$, and assume $\mathcal{F} = \{f_0, \ldots, f_{s-1}\}$. Let $\mu = \max\{|f(\mathbf{x})| \mid f \in \mathcal{F}, \mathbf{x} \in \{0,1\}^r\}$. Let $H_p = (Z_p, \binom{Z_p}{r}, \beta_p)$ denote the r-CSP-FULL$_\mathcal{F}$ instance defined as follows. For all $\mathbf{i} \in \binom{Z_p}{r}$ let

$$\beta_p(\mathbf{i}) = f_{\mathrm{col}^s_p(\mathbf{i})} \ .$$

Now fix an r-CSP$_\mathcal{F}$ instance $H = (V, E, \beta)$. The a-blowup instance $H^a = (V^a, E^a, \beta^a)$ is defined as follows: V^a is a union of families $I(i)$ of a copies of each $i \in V$, E^a is a union of the complete r-partite hypergraphs over $I(\mathbf{i}_1), \ldots, I(\mathbf{i}_r)$ for all $(\mathbf{i}_1, \ldots, \mathbf{i}_r) \in E$, and $\beta^a(\mathbf{j}) = \beta(\mathbf{i})$ whenever $\mathbf{j}_k \in I(\mathbf{i}_k)$ for $k = 1, \ldots, r$.

Claim 3.2

$$\max_\alpha \mathit{FIT}(H^a, \alpha) = a^r \max_{\alpha'} \mathit{FIT}(H, \alpha') \ .$$

To see Claim 3.2 it suffices to notice that $\max_\alpha \mathrm{FIT}(H^a, \alpha)$ is obtained when α maps $I(i)$ to one value, for all $i \in V$. We conclude from the assumption of Theorem 3.1 that it is NP-Hard to approximate the family of instances $\mathcal{I}^a = \{H^a\}_H$ of r-CSP$_\mathcal{F}$ to within an additive error of $\frac{1}{2}a^r$. In fact, it is NP-Hard to approximate r-CSP$_\mathcal{F}$ to within an additive error of $n^{r-\varepsilon}$ for any $\varepsilon > 0$, because a blowup factor a which is polynomial in $|V|$ ensures that $n^{r-\varepsilon}$ is at most $\frac{1}{2}a^r$, where $n = |V|a$ is the number of vertices of the blown-up instance.

Assume p is a prime such that $a|V| \le p \le 2a|V|$ and $s | p-1$. Such a prime exists by a known number-theoretic fact and it can be (trivially) found in polynomial (in $a|V|$) time. Now we define $H'_p = (Z_p, \binom{Z_p}{r}, \beta'_p)$ as a hybrid of H_p and H^a. The vertex set V^a is identified with a subset of Z_p. We set $\beta'_p \equiv \beta^a$ for edges $\mathbf{i} \in E^a$ and $\beta'_p \equiv \beta_p$ for all the rest.

We now claim that for all assignments $\alpha : Z_p \mapsto \{0,1\}$,

$$\left|\mathrm{FIT}(H'_p, \alpha) - \mathrm{FIT}(H^a, \alpha)\right| \le c''|V|^{3r/2} a^{r-1/2} (\log p)^{r-1} \ , \tag{5}$$

where $c'' > 0$ depends only on r, s, μ. Since the RHS of the last expression is at most $(a|V|)^{r-\varepsilon}$ for a large enough (polynomial in $|V|$), it follows from Claim 3.2 (and the conclusion thereafter) that it is it is NP-Hard to approximate $\max_\alpha \mathrm{FIT}(H'_p, \alpha)$ to within an additive error of $n^{r-\varepsilon}$ for any $\varepsilon > 0$ (where $n = p$ is the number of vertices of H'_p). Indeed, H'_p can be computed from H^a in polynomial time.

Therefore, proving Theorem 3.1 reduces to proving (5). By additivity of FIT we get

$$\begin{aligned} \left|\mathrm{FIT}(H'_p, \alpha) - \mathrm{FIT}(H^a, \alpha)\right| \le & \left|\mathrm{FIT}(H_p, \alpha)\right| \\ & + \sum_{\mathbf{k} \in E} \left|\mathrm{FIT}(H_p[I(\mathbf{k}_1), \ldots, I(\mathbf{k}_r)], \alpha)\right| \ , \end{aligned} \tag{6}$$

where $H[I(\mathbf{k}_1), \ldots, I(\mathbf{k}_r)]$ is the subgraph of H containing only edges with exactly one vertex in each of $I(\mathbf{k}_1), \ldots, I(\mathbf{k}_r)$.

We first bound $|\mathrm{FIT}(H_p, \alpha)|$. Let B_j denote $\{i \in Z_p \mid \alpha(i) = j\}$, for $j = 0, 1$. For a vector $\mathbf{b} \in \{0,1\}^r$ of bits, let $A_\mathbf{b}$ denote the set of edges

$$A_\mathbf{b} = \left\{ \mathbf{i} \in \binom{Z_p}{r} \;\middle|\; \mathbf{i}_j \in B_{\mathbf{b}_j}, j = 1, \ldots, r \right\} \ .$$

Clearly,

$$\begin{aligned} |\mathrm{FIT}(H_p, \alpha)| &= \left| \sum_{\mathbf{b}} \sum_{\mathbf{i} \in A_\mathbf{b}} \beta_p(\mathbf{i})(\mathbf{b}) \right| = \left| \sum_{\mathbf{b}} \sum_{j=0}^{s-1} n_j(A_\mathbf{b}) f_j(\mathbf{b}) \right| \\ &\le \left| \sum_{\mathbf{b}} \sum_{j=0}^{s-1} (|A_\mathbf{b}|/s) f_j(\mathbf{b}) \right| + \left| \sum_{\mathbf{b}} \sum_{j=0}^{s-1} \mu c_r p^{r/2} (r \log p)^{r-1} p^{(r-1)/2}) \right| \\ &= \left| \sum_{\mathbf{b}} \sum_{j=0}^{s-1} \mu c_r p^{r-1/2} (r \log p)^{r-1} \right| \\ &= c_r 2^r s \mu p^{r-1/2} (r \log p)^{r-1} \ . \end{aligned} \tag{7}$$

The equality of the second and third lines of (7) was by homogeneity of $\mathcal{F}$, and the inequality before is by Corollary 2.6.

Fix $\mathbf{k} \in E$. We now bound $|\mathrm{FIT}(H_p[I(\mathbf{k}_1), \ldots, I(\mathbf{k}_r)], \alpha)|$. For a permutation $\tau \in S(r)$ and bits $\mathbf{b} \in \{0,1\}^r$ Let $A_{\tau,\mathbf{b}}$ denote

$$A_{\tau,\mathbf{b}} = \left\{ \mathbf{i} \in \binom{Z_p}{r} \;\middle|\; \mathbf{i}_j \in I(\mathbf{k}_{\tau(j)}),\ \alpha(\mathbf{i}_j) = \mathbf{b}_j,\ j = 1, \ldots, r \right\} \ .$$

By definition and by Corollary 2.6,

$$\begin{aligned}
|\mathrm{FIT}(H_p[I(k_1),\ldots,I(k_r)],\alpha)| &= \left|\sum_{\tau\in S(r)}\sum_{\mathbf{b}\in\{0,1\}^r}\sum_{\mathbf{i}\in A_{\tau,\mathbf{b}}}\beta_p(\mathbf{i})(\mathbf{b})\right| \\
&= \left|\sum_{\tau\in S(r)}\sum_{\mathbf{b}\in\{0,1\}^r}\sum_{j=0}^{s-1} n_j(A_{\tau,\mathbf{b}})f_j(\mathbf{b})\right| \\
&\leq \left|\sum_{\tau\in S(r)}\sum_{\mathbf{b}\in\{0,1\}^r}\sum_{j=0}^{s-1}(|A_{\tau,\mathbf{b}}|/s)f_j(\mathbf{b})\right| \\
&\quad + \sum_{\tau\in S(r)}\sum_{\mathbf{b}\in\{0,1\}^r}\sum_{j=0}^{s-1} c_r\mu a^{r/2}(r\log p)^{r-1}p^{(r-1)/2} \\
&\leq c_r r!2^r\mu s a^{r/2} r(r\log p)^{r-1}p^{(r-1)/2}
\end{aligned} \tag{8}$$

Plugging (7) and (8) in (6), we get

$$|\mathrm{FIT}(H_p',\alpha)-\mathrm{FIT}(H^a,\alpha)| \leq c'(\log p)^{r-1}(p^{r-1/2}+|E|a^{r/2}p^{(r-1)/2}) \ ,$$

where $c'>0$ depends only on r,s,μ. Since $p=\Theta(|V|a)$, $|E|\leq|V|^r$, we conclude that

$$|\mathrm{FIT}(H_p',\alpha)-\mathrm{FIT}(H^a,\alpha)| \leq c''|V|^{3r/2}a^{r-1/2}(\log p)^{r-1}, \tag{9}$$

where $c''>0$ depends only on r,s,μ, as desired.

Corollary 3.3 *1. It is NP-Hard to approximate dense MAX-r-LIN-2 for $r\geq 2$ to within an additive error of $n^{r-\varepsilon}$ for any $\varepsilon>0$.*

2. Is is NP-Hard to approximate dense MAX-r-CNF to within an additive factor of $n^{r-\varepsilon}$ for any $\varepsilon>0$.

Proof: To prove the first statement, take $\mathcal{F}=\{f_0,f_1\}$, where for $\mathbf{x}\in Z_2^r$

$$f_0(\mathbf{x}) = \begin{cases} 1/2 & \sum_{i=1}^r \mathbf{x}_i = 0 \\ -1/2 & \text{otherwise} \end{cases}$$

$$f_1(\mathbf{x}) = \begin{cases} 1/2 & \sum_{i=1}^r \mathbf{x}_i = 1 \\ -1/2 & \text{otherwise} \end{cases}$$

It is clear that $\mathcal{F}$ is homogenous. It is also clear that computing r-CSP$_{\mathcal{F}}$ to within an additive error of 1 is equivalent to solving MAX-r-LIN-2. Since MAX-r-LIN-2 is NP-Hard, we conclude by Theorem 3.1 that it is NP-Hard to approximate fully dense MAX-r-LIN-2 to within an additive error of $n^{r-\varepsilon}$, for all $\varepsilon>0$.

To prove the second statement, take $\mathcal{F}=\{f_0,\ldots,f_{2^r-1}\}$, where each f_b corresponds to a CNF clause $\varphi(b)$ on r variables, where the i'th variable is negated if and only if the i'th bit of b in binary representation is 1. The value $f(\mathbf{x})$ of $\mathbf{x}\in\{0,1\}^r$ is 2^{-r} if x satisfies $\varphi(b)$, otherwise $2^{-r}-1$. Clearly, $\mathcal{F}$ is homogenous, and computing MAX-r-CNF to within an additive error of 1 is equivalent to solving the NP-Hard MAX-r-CNF. We conclude that it is NP-Hard to approximate fully dense MAX-r-CNF to within an additive error of $n^{r-\varepsilon}$ for any $\varepsilon>0$.

4 r-CSP-RANK

An r-CSP-RANK$_{\mathcal{F}}$ instance consists of the following elements.

1. A finite family $\mathcal{F}$ of constraints $f : S(r) \to \mathbb{R}$, where r is a constant.
2. An r-uniform hypergraph $H = (V, E, \beta)$ with vertex set V and edge set E, equipped with a function $\beta : E \to \mathcal{F}$.

We will assume that $r, \mathcal{F}$ are fixed and do not form part of the input. For a ranking π of V, we associate an objective function $\mathrm{FIT}(H, \pi)$, defined as

$$\mathrm{FIT}(H, \pi) = \sum_{\mathbf{i} \in E} \beta(\mathbf{i})(\mathrm{ord}_\pi(\mathbf{i})) \ ,$$

where $\mathrm{ord}_\pi(\mathbf{i}) \in S(r)$ is the internal ranking of $\mathbf{i}_1, \dots, \mathbf{i}_r$ induced by π. In the last sum, we assume as usual that $\mathbf{i}_1 < \cdots < \mathbf{i}_r$ for canonization. We say that $\mathcal{F}$ is *homogenous* if for any fixed $\sigma \in S(r)$,

$$\sum_{f \in F} f(\sigma) = 0 \ .$$

We will assume in what follows that $\mathcal{F}$ is *always* homogenous.

Let r-CSP-RANK-FULL$_{\mathcal{F}}$ denote the problem r-CSP-RANK$_{\mathcal{F}}$ restricted to fully dense instances, that is, instances in which $E = \binom{V}{r}$.

Our main result is as follows.

Theorem 4.1 *If r-CSP-RANK$_{\mathcal{F}}$ is NP-Hard to approximate to within an additive error of* 1 *for some fixed $r, \mathcal{F}$, then r-CSP-RANK-FULL$_{\mathcal{F}}$ is NP-Hard to approximate to within an additive factor of $n^{r-\varepsilon}$ for any $\varepsilon > 0$, where n is the number of vertices in the hypergraph.*

We sketch the proof, which is very similar to that of Theorem 3.1. Let $\mu = \max\{|f(\sigma)| \mid f \in \mathcal{F},\ \sigma \in S(r)\}$. We define an r-CSP-RANK-FULL$_{\mathcal{F}}$ instance $H_p = (Z_p, \binom{Z_p}{r}, \beta_p)$ for a prime p such that $s|p-1$, where $s = |\mathcal{F}|$. Let $\mathcal{F} = \{f_1, \dots, f_s\}$. We set $\beta_p(\mathbf{i}) = f_{\mathrm{col}_p^s[\mathbf{i}]}$ for $\mathbf{i} \in \binom{Z_p}{r}$.

Now, given an instance H of r-CSP-RANK$_{\mathcal{F}}$, we reduce to an instance H'_p, which is a hybrid of a blow-up H^a of H and H_p, where a will be chosen later and $a|V| \le p \le 2a|V|$, $s|p-1$, exactly as we did in Section 3. We claim that

Claim 4.2

$$\max_\alpha FIT(H^a, \pi) = a^r \max_{\pi'} FIT(H, \pi') \ .$$

To see claim 4.2 it suffices to notice that $\max_\pi \mathrm{FIT}(H^a, \pi)$ is obtained when all vertices of each $I(i)$ form a consecutive block with respect to the order π, for all $i \in V$. We conclude by the assumption of Theorem 4.1 that it is NP-Hard to approximate the family of instances $\mathcal{I}^a = \{H^a\}_H$ of r-CSP-RANK$_{\mathcal{F}}$ to within an additive error of $\frac{1}{2}a^r$. In fact, it is NP-Hard to approximate r-CSP-RANK$_{\mathcal{F}}$ to within an additive error of $n^{r-\varepsilon}$ for any $\varepsilon > 0$, because a blowup factor a which is polynomial in $|V|$ ensures that $n^{r-\varepsilon}$ is at most $\frac{1}{2}a^r$, where $n = |V|a$ is the number of vertices of the blown instance.

We need the following technical lemma.

Lemma 4.3 *Let $B'_1, \ldots, B'_r \subseteq Z_p$. Fix $\pi \in S(p), \sigma \in S(r)$. Let $A' = A'(B'_1, \ldots, B'_r) \subseteq \binom{Z_p}{r}$ denote the set*

$$A' = \left\{ \mathbf{i} \in \binom{Z_p}{r} \;\middle|\; ord_\pi(\mathbf{i}) = \sigma,\ \mathbf{i}_j \in B'_j,\ j = 1, \ldots, r \right\} .$$

Then

$$\left| n_j(A') - |A'|/s \right| \le \tilde{c}_r (|B_1| \cdots |B_r|)^{1/2} (r \log p)^{2r-2} p^{(r-1)/2} , \tag{10}$$

for some $\tilde{c}_r > 0$ which depends only on r.

The proof of Lemma 4.3 is similar to that of Lemma 2.5 and Corollary 2.6. We decompose A' to a family of classes $\mathcal{S}'(B'_1, \ldots, B'_r)$ with the following change: We take k such that $|B'_k| \ge |B'_j|$ for $j = 1, \ldots, r$, and instead of halving B'_k according to the order $<$, we half it according to the order $<_\pi$. Each $S' \in \mathcal{S}'(B'_1, \ldots, B'_r)$ is a product $S'_1 \times \cdots S'_r$ such that $S'_1 <_\pi \cdots <_\pi S'_r$, $S'_j \subseteq B'_j$. This gives rise to a subset of A', namely $A = A(S'_{\sigma(1)}, \ldots, S'_{\sigma(r)})$ (using the notation of Lemma 2.5). Then we use Corollary 2.6 to bound $|n_j(A) - |A|/s|$. We arrive at a recursive formula almost identical to that in Lemma 2.5 which solves to a function bounded by the RHS of (10).

To complete the proof of Theorem 4.1, it suffices to show that for all $\pi \in S(p)$,

$$\left| \mathrm{FIT}(H'_p, \pi) - \mathrm{FIT}(H^a, \pi) \right| \le c'' |V|^{3r/2} a^{r-1/2} (\log p)^{2r-2} ,$$

for some global $c'' > 0$ that depends only on r, s, μ. By additivity of FIT, we get

$$\begin{aligned} |\mathrm{FIT}(H'_p, \pi) - \mathrm{FIT}(H^a, \pi)| \le{}& |FIT(H_p, \pi)| \\ &+ \sum_{\mathbf{k} \in E} |\mathrm{FIT}(H_p[I(\mathbf{k}_1), \ldots, I(\mathbf{k}_r)], \pi)| , \end{aligned} \tag{11}$$

where $H_p[I(\mathbf{k}_1), \ldots, I(\mathbf{k}_r)]$ is the subgraph of H containing only edges with exactly one vertex in each of $I(\mathbf{k}_1), \ldots, I(\mathbf{k}_r)$. For a ranking $\sigma \in S(r)$ let

$$A_\sigma = \left\{ \mathbf{i} \in \binom{Z_p}{r} \;\middle|\; \mathrm{ord}_\pi(\mathbf{i}) = \sigma \right\} .$$

$$\begin{aligned} |\mathrm{FIT}(H_p, \pi)| &= \left| \sum_{\sigma \in S(r)} \sum_{\mathbf{i} \in A_\sigma} \beta_p(\mathbf{i})(\sigma) \right| = \left| \sum_{\sigma \in S(r)} \sum_{j=0}^{s-1} n_j(A_\sigma) f_j(\sigma) \right| \\ &\le \left| \sum_{\sigma} \sum_{j=0}^{s-1} (|A_\sigma|/s) f_j(\sigma) \right| + \left| \sum_{\sigma} \sum_{j=0}^{s-1} \mu \tilde{c}_r p^{r/2} (r \log p)^{2r-2} p^{(r-1)/2} \right| \\ &= \left| \sum_{\sigma} \sum_{j=0}^{s-1} \mu \tilde{c}_r p^{r-1/2} (r \log p)^{2r-2} \right| \\ &= c_r r! s \mu p^{r-1/2} (r \log p)^{2r-2} . \end{aligned} \tag{12}$$

The equality between the second and the third lines are by homogeneity of $\mathcal{F}$, and the inequality before is by Lemma 4.3.

Fix $\mathbf{k} \in E$. We now bound $|\mathrm{FIT}(H_p[I(\mathbf{k}_1), \ldots, I(\mathbf{k}_r)], \pi)|$. For permutations $\tau, \sigma \in S(r)$, let $A_{\tau,\sigma}$ denote

$$A_{\tau,\sigma} = \left\{ \mathbf{i} \in \binom{Z_p}{r} \;\middle|\; \mathbf{i}_j \in I(\mathbf{k}_{\tau(j)}),\ \mathrm{ord}_\pi(\mathbf{i}) = \sigma \right\} .$$

By definition and by Lemma 4.3,

$$\begin{aligned}|\text{FIT}(H_p[I(k_1),\ldots,I(k_r)],\pi)| &= \left|\sum_{\tau\in S(r)}\sum_{\sigma\in S(r)}\sum_{\mathbf{i}\in A_{\tau,\sigma}}\beta_p(\mathbf{i})(\sigma)\right| \\ &= \left|\sum_{\tau\in S(r)}\sum_{\sigma\in S(r)}\sum_{j=0}^{s-1} n_j(A_{\tau,\sigma})f_j(\sigma)\right| \\ &\le \left|\sum_{\tau\in S(r)}\sum_{\sigma\in S(r)}\sum_{j=0}^{s-1}(|A_{\tau,\sigma}|/s)f_j(\sigma)\right| \\ &\quad + \sum_{\tau\in S(r)}\sum_{\sigma\in S(r)}\sum_{j=0}^{s-1} c_r\mu a^{r/2}(r\log p)^{2r-2}p^{(r-1)/2} \\ &\le c_r(r!)^2\mu s a^{r/2}(r\log p)^{2r-2}p^{(r-1)/2}\end{aligned} \tag{13}$$

Plugging (12) and (13) in (11), we get

$$|\text{FIT}(H'_p,\pi) - \text{FIT}(H^a,\pi)| \le c'(\log p)^{2r-2}(p^{r-1/2} + |E|a^{r/2}p^{(r-1)/2}) \ ,$$

where $c' > 0$ depends only on r, s, μ. Since $p = \Theta(|V|a), |E| \le |V|^r$, we conclude that

$$|\text{FIT}(H'_p,\pi) - \text{FIT}(H^a,\pi)| \le c''|V|^{3r/2}a^{r-1/2}(\log p)^{2r-2}, \tag{14}$$

where $c'' > 0$ depends only on r, s, μ as desired.

Corollary 4.4

- *It is NP-Hard to approximate FEEDBACK-ARC-SET-TOUR to within an additive error of $n^{2-\varepsilon}$, for any $\varepsilon > 0$. (This special case was proven by Alon in [3].)*
- *It is NP-Hard to approximate BETWEENNESS-TOUR to within an additive error of $n^{3-\varepsilon}$, for any $\varepsilon > 0$.*

Proof: To prove the first statement set $r = 2$, $s = 2$, $\mathcal{F} = \{f_1, f_2\}$, with $f_1((1\ 2)) = 1/2$, $f_1((2\ 1)) = -1/2$, $f_2((1\ 2)) = -1/2$, $f_2((2\ 1)) = 1/2$. Clearly $\mathcal{F}$ is homogenous, and it is also clear that computing r-CSP-RANK$_{\mathcal{F}}$ to within an additive error of 1 is equivalet to solving FEEDBACK-ARC-SET exactly. Since FEEDBACK-ARC-SET is NP-Hard, we conclude by Theorem 4.1 that it is NP-Hard to approximate FEEDBACK-ARC-SET-TOUR to within an additive error of $n^{2-\varepsilon}$ for any $\varepsilon > 0$.

To prove the second statement, set $r = 3$, $s = 3$, $\mathcal{F} = \{f_1, f_2, f_3\}$, where for $\sigma \in S(3)$, $f_j(\sigma)$ equals $2/3$ if j is ranked between $\{1, 2, 3\} - \{j\}$ by σ, otherwise $-1/3$. Clearly $\mathcal{F}$ is homogenous, and computing r-CSP-RANK$_{\mathcal{F}}$ to within an additive error of 1 is equivalent to solving BETWEENNESS exactly. Therefore, we conclude by Theorem 4.1 that it is NP-Hard to approximate BETWEENNESS-TOUR to within an additive error of $n^{3-\varepsilon}$ for any $\varepsilon > 0$. Note that by known results (e.g. [8]) it is possible to approximate BETWEENNESS-TOUR to within an additive error of εn^3 for any $\varepsilon > 0$ in polynomial time.

5 Open Problems and Remarks

- The techniques introduced here break down when trying to prove hardness of approximation with respect to a *multiplicative* error. For example, it is not known if there is a PTAS for

FEEDBACK-ARC-SET-TOUR. It would be interesting to extend the methods in a way that will enable one to tackle this problem.

- A property of graphs is called monotone if it is closed under omitting vertices and edges. It is dense if there are n vertex graphs with $\Omega(n^2)$ edges satisfying it. Thus, for example, the property of being bipartite, being k-colorable, being triangle-free or containing no subgraph with 6 vertices and at least 10 edges are all monotone and dense. In [6] the authors combine techniques similar to the ones used in [1] and [3] with ideas from extremal graph theory to show that for every monotone, dense graph property P and for any fixed $\varepsilon > 0$, it is NP-hard to approximate within an additive error of $n^{2-\varepsilon}$ the minimum number of edges that have to be deleted from a given n-vertex graph to get a graph that satisfies P. It seems plausible that a similar result holds for hyergraphs, but a proof will require some additional ideas.

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