Fair representation by independent sets

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Abstract

Given a partition of a set V into pairwise disjoint sets $\mathcal{V} = (V_1, V_2, \ldots, V_m)$ and a number $\alpha \leqslant 1$, a subset S of $V := \bigcup \mathcal{V}$ is said to *represent* V_i α*-fairly* if $|S \cap V_i| \geqslant \alpha |V_i|$, and it is said to *represent* $\mathcal{V}$ α*-fairly* if it represents α-fairly all V_is. We wish to represent nearly fairly (the meaning of "nearly" will transpire below) the sets V_i with large α, by a set S of vertices that are independent in a given graph on V. We study the following two conjectures:

Conjecture 1. *Suppose that the edges of $K_{n,n}$ are partitioned into sets $E_1, E_2, \ldots, E_m$. Then there exists a perfect matching F in $K_{n,n}$ satisfying $|F \cap E_i| \geqslant \left\lfloor \frac{|E_i|}{n} \right\rfloor - 1$, with strict inequality holding for all but at most one value of i.*

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and:

Conjecture 2. *If P is a path and $V = V(F) = V_1 \cup \ldots \cup V_m$ is a partition, then there exists a subset S of V that is independent in P, and satisfies $|S| \geqslant \frac{|V|-m}{2}$, and $|S \cap V_i| \geqslant \frac{|V_i|}{2} - 1$ for all $i \leqslant m$.*

These conjectures extend and modify several well studied questions. We prove the existence of sets satisfying either condition of Conjecture 2 (not necessarily both), and we prove Conjecture 1 for $m = 2$ and $m = 3$. The proofs are in part topological, using Sperner's lemma, the Borsuk-Ulam theorem and a theorem of Schrijver on subgraphs of Kneser's graph that are critical for the chromatic number.

Keywords: matchings, independent sets, fair representation.

1 Introduction: A conjecture of Ryser and ramifications

Of the many directions from which the subject of this paper can be approached, let us introduce it through the lens of an attractive conjecture - Ryser's Latin squares conjecture. Given an $n \times n$ array A of symbols, a *partial transversal* is a set of entries taken from distinct rows and columns, and containing distinct symbols. A partial transversal of size n is called simply a *transversal*. Ryser's conjecture [27] is that if A is a Latin square, and n is odd, then A necessarily has a transversal. The oddness condition is indeed necessary - for every even $n > 0$ there exist $n \times n$ Latin squares not possessing a transversal. An example is the addition table of $\mathbb{Z}_n$: if a transversal T existed for this Latin square, then the sum of its elements, modulo n, is $\sum_{k \leqslant n} k = \frac{n(n+1)}{2} (\text{mod} n)$. On the other hand, since every row and every column is represented in this sum, the sum is equal to $\sum_{i \leqslant n} i + \sum_{j \leqslant n} j = n(n+1)(\text{mod} n)$, and for n even the two results do not agree. Arsovski [14] proved a closely related conjecture, of Snevily, that every square submatrix (whether even or odd) of the addition table of an odd order abelian group possesses a transversal.

Brualdi [17] and Stein [32] conjectured that for any n, any Latin square of order n has a partial transversal of order $n-1$. Stein [32] observed that the same conclusion may follow from weaker conditions - the square does not have to be Latin. Possibly it is enough to assume that the entries of the $n \times n$ square are equally distributed between n symbols. Put in the terminology of bipartite graphs, this reads:

Conjecture 1.1. *If the edge set of $K_{n,n}$ is partitioned into sets $E_1, E_2, \ldots, E_n$ of size n each, then there exists an almost perfect matching in $K_{n,n}$ consisting of one edge from all but possibly one E_i.*

In this conjecture there is no distinction between n even and odd. It is easy to construct examples of squares satisfying Stein's condition, in which there is no full transversal for n even as well as for n odd. In matrix language, take a matrix M with $m_{i,j} = i$ for $j < n$, and $m_{i,n} = i + 1(\text{mod} n)$ (in particular, $m_{n,n} = 1$).

This conjecture belongs to a wider family of problems. Given a complex (closed down hypergraph) $\mathcal{C}$ on a vertex set V, and a partition $\mathcal{V}$ of V into sets $V_1, V_2, \ldots, V_m$, we may ask for the largest number α for which there exists a subset S of V belonging to $\mathcal{C}$, and satisfying $|S \cap V_i| \geqslant \lfloor \alpha |V_i| \rfloor$ for all i, or, as in Stein's conjecture, for all values of i but one, or for all i but a fixed number.

A tight bound for α can be obtained in the case of matroids. For a given complex $\mathcal{C}$, let $\beta(\mathcal{C})$ be the minimal number of edges (simplices) of $\mathcal{C}$ whose union is $V(\mathcal{C})$. A result following directly from Edmonds' matroid intersection theorem is:

Theorem 1.2. *If $\mathcal{C}$ is a matroid then for every partition $\mathcal{V}$ of $V(\mathcal{C})$ there exists a set $S \in \mathcal{C}$ satisfying $|S \cap V_i| \geqslant \left\lfloor \frac{|V_i|}{\beta(\mathcal{C})} \right\rfloor$ for all i.*

We shall mainly be interested in the case that $\mathcal{C}$ is the complex of independent sets of a graph G, denoted by $\mathcal{I}(G)$. The following theorem of Haxell [20] pinpoints the right value of α in this case:

Theorem 1.3. *If $\mathcal{V} = (V_1, V_2, \ldots, V_m)$ is a partition of the vertex set of a graph G, and if $|V_i| \geqslant 2\Delta(G)$ for all $i \leqslant m$, then there exists a set S independent in G, intersecting all V_is.*

This was an improvement over earlier results of Alon, who proved the same with $25\Delta(G)$ [10] and then with $2e\Delta(G)$ [11].

Corollary 1.4. *If the vertex set V of a graph G is partitioned into sets $V_1, V_2, \ldots, V_m$ then there exists an independent subset S of V, satisfying $|S \cap V_i| \geqslant \left\lfloor \frac{|V_i|}{2\Delta(G)} \right\rfloor$ for every $i \leqslant m$.*

Proof. For each $i \leqslant m$ pack $\left\lfloor \frac{|V_i|}{2\Delta(G)} \right\rfloor$ disjoint sets of size $2\Delta(G)$ (call them V_i^j) in each V_i. By Theorem 1.3 there exists an independent set S meeting all V_i^j, and this is the set desired in the theorem. $\square$

So, for $\mathcal{C} = \mathcal{I}(G)$, the complex of independent sets in G, the magic number is $\alpha = \frac{1}{2\Delta}$. This is sharp, as shown in [35, 21, 33]. Note that $\frac{1}{2\Delta} \leqslant \frac{1}{\beta(\mathcal{I}(G))} = \frac{1}{\chi(G)}$, with strict inequality holding in general, meaning that $\mathcal{I}(G)$ does not behave so well with respect to representation as matroids do.

The secret connecting this result to Stein's conjecture and to Conjecture 1.1 is that for line graphs much better bounds can be obtained.

Theorem 1.5. *If H is a graph and $G = L(H)$ (the line graph of H) then there exists an independent set S such that $|S \cap V_i| \geqslant \left\lfloor \frac{|V_i|}{\Delta(G)+2} \right\rfloor$ for every $i \leqslant m$.*

This follows from a result, proved in [3], that if G is a line graph of a graph then the topological connectivity of $\mathcal{I}(G)$, denoted by $\eta(\mathcal{I}(G))$, satisfies

$$\eta(\mathcal{I}(G)) \geqslant \frac{|V|}{\Delta(G)+2} \tag{1}$$

In [7] a generalization of (1) was proved for hypergraphs:

$$\text{If } H \text{ is a hypergraph and } G = L(H) \text{ then } \eta(\mathcal{I}(G)) \geqslant \frac{|V|}{\max_{e\in H} \sum_{v\in e} deg_H(v)}. \tag{2}$$

To see why (2) generalizes (1) note that if H is r-uniform and linear (no two edges meet at more than one vertex), then $\Delta(G) = \max_{e\in H} \sum_{v\in e} deg_H(v) - r$, which together with (2) entails that $\eta(\mathcal{I}(G)) \geqslant \frac{|V|}{\Delta(G)+r}$.

The connectivity $\eta(\mathcal{C})$ of a complex $\mathcal{C}$ is the minimal dimension of a hole in $\mathcal{C}$, so for example if $\mathcal{C}$ has a non-empty vertex set but is not path connected then $\eta(\mathcal{C}) = 1$, since there is a "hole" of dimension 1, consisting of two points that cannot be joined by a path (it is this non existing path that is the hole of dimension 1). The way from (1) to Theorem 1.5 goes through a topological version of Hall's theorem, proved in [6].

In an even stronger version of Conjecture 1.1, the single E_i that is not represented can be arbitrarily chosen. To put this formally, we shall use the following terminology:

Definition 1.6. A *rainbow set* of a collection $S_1, S_2, \ldots, S_m$ is a choice function from these sets. If S_i are sets of graph edges and the chosen edges form a matching, then the rainbow set is called a *rainbow matching.*

Conjecture 1.7. *Any family $E_1, E_2, \ldots, E_{n-1}$ of disjoint subsets of size n of $E(K_{n,n})$ has a rainbow matching.*

The three conditions - that the sets E_i are disjoint, that they are subsets of $E(K_{n,n})$, and that their number is $n-1$, seem a bit artificial. It is enticing to make the following conjecture, that entails the case in which the sets E_i are matchings:

If $E_1, E_2, \ldots, E_m$ are sets of edges in a bipartite graph, and $|E_i| > \Delta(\bigcup_{i \leqslant m} E_i)$ (where $\Delta(\bigcup_{i \leqslant m} E_i)$ is the maximal degree of a vertex in the multigraph $\bigcup_{i \leqslant m} E_i$) then there exists a rainbow matching.

Unfortunately, this conjecture is false, as shown by the following example:

Example 1.8. [21, 35] Take three vertex disjoint copies of C_4, say A_1, A_2, A_3. Number the edges of A_i cyclically as a_i^j $(j = 1 \ldots 4)$. Let $E_1 = \{a_1^1, a_1^3, a_3^1\}$, $E_2 = \{a_1^2, a_1^4, a_3^3\}$, $E_3 = \{a_2^1, a_2^3, a_3^2\}$ and $E_4 = \{a_2^2, a_2^4, a_3^4\}$. Then $\Delta(\bigcup_{i \leqslant m} E_i) = 2$, $|E_i| = 3$ and there is no rainbow matching.

In [4] the following was suggested:

Conjecture 1.9. *If $E_1, E_2, \ldots, E_m$ are sets of edges in a bipartite graph, and $|E_i| > \Delta(\bigcup_{i \leqslant m} E_i) + 1$ then there exists a rainbow matching.*

Re-phrased, this conjecture reads: If H is a bipartite multigraph, $G = L(H)$ and $V_i \subseteq V(G)$ satisfy $|V_i| \geqslant \Delta(H) + 2$ for all i, then there exists an independent set in G (namely a matching in H) meeting all V_is. We do not know of other examples, beyond Example 1.8, in which $|V_i| \geqslant \Delta(H) + 1$ does not suffice. The conjecture is false if the sets V_i are allowed to be multisets - we omit the details of the example showing this.

Conjecture 1.9 would yield:

Conjecture 1.10. *If the edge set of a graph H is partitioned into sets $F_1, \ldots, F_m$ then there exists a matching M satisfying $|M \cap F_i| \geqslant \left\lfloor \frac{|F_i|}{\Delta(H)+2} \right\rfloor$ for all $i \leqslant m$*

If true, Conjecture 1.7 would imply:

Conjecture 1.11. *Suppose that $E(K_{n,n})$ is partitioned into sets $E_1, E_2, \ldots, E_m$. Then there exists a perfect matching F in $K_{n,n}$ satisfying $|F \cap E_i| \geqslant \left\lfloor \frac{|E_i|}{n} \right\rfloor - 1$, with strict inequality holding for all but one value of i.*

This can be strengthened to:

Conjecture 1.12. *In Conjecture 1.11 it is possible to choose the index for which the strict inequality does not occur. Namely, for every $j \leqslant m$ there exists a perfect matching F in $K_{n,n}$ satisfying $|F \cap E_i| \geqslant \left\lfloor \frac{|E_i|}{n} \right\rfloor$ for all $i \neq j$, and $|F \cap E_j| \geqslant \left\lfloor \frac{|E_j|}{n} \right\rfloor - 1$.*

We shall prove:

Theorem 1.13. *Conjecture 1.12 is true for $m = 2, 3$, for all n.*

For $m = 2$ we shall also characterize the cases where (-1) is needed for one of the indices.

Line graphs are one family of graphs where Theorem 1.2 can be improved upon. Another is paths.

Conjecture 1.14. *Given a path P and a partition of $V(P)$ into sets $V_1, \ldots, V_m$ there exists an independent set S such that $|S \cap V_i| \geqslant \frac{|V_i|}{2} - 1$ for all i, with strict inequality holding for all but at most $\frac{m}{2}$ sets V_i.*

While the full conjecture is open, we shall prove the existence of an independent set satisfying either one of the two conditions.

Theorem 1.15. *Given a partition of the vertex set of a path into sets $V_1, \ldots, V_m$ there exists an independent set S and integers b_i, $i \leqslant m$, such that $\sum_{i \leqslant m} b_i \leqslant \frac{m-1}{2}$ and $|S \cap V_i| \geqslant \frac{|V_i|}{2} - b_i$ for all i.*

The second result we shall prove applies also to cycles. Of course, Theorem 1.15 can be proved also for cycles, up to one vertex:

Theorem 1.16. *Given a partition of the vertex set of a cycle into sets $V_1, \ldots, V_m$ there exists an independent set S such that $|S \cap V_i| \geqslant \frac{|V_i|}{2} - 1$ for all i.*

The proofs of three of our main results - that of Theorem 1.13 for the case $m = 3$, and those of Theorems 1.15 and 1.16, are topological. The first uses Sperner's lemma, and the second uses the Borsuk-Ulam theorem.

The proof of Theorem 1.16 uses a theorem of Schrijver, strengthening a famous theorem of Lovász on the chromatic number of Kneser graphs. This means that it, too, uses the Borsuk-Ulam theorem, since the Lovász-Schrijver proof uses the latter. We refer the reader to Matousek's book [25] for background on topological methods in combinatorics, in particular the use of the Borsuk-Ulam theorem.

2 Fair representation by independent sets in paths: a Borsuk-Ulam approach

In this section we prove Theorem 1.15. Following an idea from the proof of the "necklace theorem" [9], we shall use the Borsuk-Ulam theorem. In the necklace problem two thieves want to divide a necklace with m types of beads, each recurring in an even number of beads, so that the beads of every type are evenly split between the two. The theorem is that the thieves can achieve this goal using at most m cuts of the necklace. In our case, the two "thieves" are the sets of odd and even points, respectively. But rather than use the theorem as a black box, we have to adapt ideas from its proof to the present situation.

Proof of Theorem 1.15

Let $v_1, \ldots, v_n$ be the vertices of P_n, ordered along the path. Our aim is to form an independent set meeting each of the sets V_i partitioning V in approximately $|V_i|/2$ vertices. In order to use the Borsuk-Ulam theorem, we first make the problem continuous, by replacing each v_p by the characteristic function of the pth of n intervals of length $\frac{1}{n}$ in $[0,1]$, namely by $\tilde{v}_p = \chi_{[\frac{p-1}{n}, \frac{p}{n}]}$. We call the interval $[\frac{p-1}{n}, \frac{p}{n}]$ a *bead*.

As usual, S^{m-1} denotes the set of points $\vec{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ satisfying $\sum_{i \leqslant m} x_i^2 = 1$. Given such a point $\vec{x}$, let $z_k = \sum_{j \leqslant k} x_j^2$ ($z_0 = 0$).

Let g be the characteristic function of the odd beads on the path. Explicitly, for every $1 \leqslant p \leqslant n$ odd let $g(y) = 1$ for all $\frac{p-1}{n} \leqslant y \leqslant \frac{p}{n}$ and for every $1 \leqslant p \leqslant n$ even let $g(y) = 0$ for all $\frac{p-1}{n} \leqslant y \leqslant \frac{p}{n}$. Let h be the characteristic function of the even beads, namely $h(y) = 1 - g(y)$.

For every $i \leqslant m$ let χ_i be the sum of all characteristic functions of beads belonging to V_i. That is, for every $0 \leqslant p \leqslant n-1$ and all $\frac{p-1}{n} \leqslant y \leqslant \frac{p}{n}$ let $\chi_i(y) = 1$ if $v_p \in V_i$ and $\chi_i(y) = 0$ otherwise.

For every $i \leqslant m$ define a function $f_i: \ S^{m-1} \to \mathbb{R}^{m-1}$ by:

$$f_i(x_1, \ldots, x_m) = \sum_{1 \leqslant k \leqslant m} \int_{z_{k-1}}^{z_k} (g(y) - h(y))\chi_i(y)\mathrm{sign}(x_k)dy$$

Here, as usual, $\mathrm{sign}(x) = 0$ if $x = 0$ $\mathrm{sign}(x) = 1$ if $x > 0$ and $\mathrm{sign}(x) = -1$ if $x < 0$.

The functions f_i are continuous, because at the points of discontinuity of the sign function the intervals in the integrals are of zero length. The sign term guarantees that $f_i(-\vec{x}) = -f_i(\vec{x})$. Hence, by the Borsuk-Ulam theorem there exists a point $\vec{w} = (w_1, \ldots, w_m) \in S^{m-1}$ such that $f_i(\vec{w}) = 0$ for all $i \in [m]$, where $z_k = \sum_{j \leqslant k} w_j^2$.

For $y \in [0,1]$ such that $y \in [z_{k-1}, z_k)$ define $POS(y) = 1$ if $w_k \geqslant 0$ and $POS(y) = 0$ otherwise. Let $NEG(y) = 1 - POS(y)$. Let $v(y) = POS(y)g(y) + NEG(y)h(y)$. So, v chooses the odd beads (=vertices) from intervals on which $\vec{w}$ is positive, and the even beads from intervals on which $\vec{w}$ is negative. This choice does not necessarily constitute an independent set of vertices - the desired independent set I will be a subset of this choice.

Fix now $i \in [m]$. The fact that $f_i(\vec{w}) = 0$ means that

$$\int_{y=0}^{1} \chi_i(y) POS(y)[g(y) - h(y)]dy = \int_{y=0}^{1} \chi_i(y) NEG(y)[g(y) - h(y)]dy$$

Shuffling terms this gives:

$$\int_{y=0}^{1} \chi_i(y)[POS(y)g(y) + NEG(y)h(y)]dy = \int_{y=0}^{1} \chi_i(y)[POS(y)h(y) + NEG(y)g(y)]dy$$

On the left hand side there is the measure of the set of points of V_i chosen by v, and on the roght hand side the measure of the set of points of V_i not chosen by v. So,

$$\int_{y=0}^{1} \chi_i(y)[POS(y)g(y) + NEG(y)h(y)]dy = \frac{1}{2} \int_{y=0}^{1} \chi_i(y)dy$$

.

Let F be the set of beads containing a point z_k for some k. Let I_1 be the set of those odd beads that meet intervals $(z_{k-1}, z_k]$ in which $w_k \geqslant 0$ and even beads that meet intervals $(z_{k-1}, z_k]$ in which $w_k < 0$. Let I_2 be the set of those even beads that meet intervals $(z_{k-1}, z_k]$ in which $w_k \geqslant 0$ and odd beads that meet intervals $(z_{k-1}, z_k]$ in which $w_k < 0$. Note that the two may overlap at beads containing points z_k. We shall choose as I either $I_1 \setminus F$ or $I_2 \setminus F$. Let us show that for one of these choices $|I| \geqslant \frac{n}{2} - \frac{m-1}{2}$. Note that in passing from the beads (living in $[0,1]$) to the sets I_j (living in $[n]$) quantities should be multiplied by n

Denote by α_j^k the "losses" incurred by the beads belonging to I_j upon renouncing the beads containing z_k. Then, clearly, $\alpha_1^k + \alpha_2^k = \frac{m-1}{n}$. With the n-factor involved in passing to the sizes of I_j, it follows that $|I_1 \setminus F| + |I_2 \setminus F| \geqslant n - (m-1)$. Taking I as the larger of $I_1 \setminus F, \ \ I_2 \setminus F$, I satisfies the conditions of the theorem. □

Remark 2.1.

1. We do not know whether it is always possible to divide the losses evenly between the various V_is. In particular, we do not know whether in fact all b_is in the theorem can assume only the values 0 or 1.

2. The inequality $\sum_{i \leqslant m} b_i \leqslant \frac{m-1}{2}$ cannot be improved. Namely, there are examples in which the minimum of the sum $\sum_{i \leqslant m} b_i$ in the theorem is $\frac{m-1}{2}$. To see this, let $m = 2k+1$, and let each V_i be of size $2k$. Consider a sequence of length $2k \times (2k+1)$, in which the $(i-1)m + 2j - 1$-th element belongs to V_i $(i = 1, \ldots, 2k, \ \ j = 1, \ldots, k+1)$, and the rest of the elements are chosen in any way so as to satisfy the condition $|V_i| = 2k$. For example, if $k = 2$ then the sequence is of the form:

$$1 * 1 * 1 - 2 * 2 * 2 - 3 * 3 * 3 - 4 * 4 * 4$$

where the $*$s can be filled in any way that satisfies $|V_i| = 4$ (namely, four of them are replaced by the symbol 5, and one is replaced by i for each symbol $i = 1, 2, 3, 4$), and the dashes are just for ease of reference to the four stretches. If S is an independent set in the path then, removing occurrences of symbol i so as to have at most k occurrences of i (for example, in the first stretch of the example above there is no point in choosing all three 1s), we may assume that S contains no more than k elements from each stretch, and thus $|S| \leqslant 2k \times k$, which is $\frac{m-1}{2}$ short of half the length of the path.

3. It may be of interest to find the best bounds as a function of the sizes of the sets V_i and their number. Note that in the example above the size of the sets is almost equal to their number. As one example, if If all V_is are of size 2, then the inequality can be improved to: $\sum_{i \leqslant m} b_i \leqslant \frac{m}{3}$. To see this, look at the multigraph obtained by adding to P_n the pairs forming the sets V_i as edges. In the resulting graph the maximum degree is 3, and hence by Brooks' theorem it is 3-colorable (in fact, the theorem needs not be invoked: the average degree in every induced subgraph is less than 3). Thus there is an independent set of size at least $\frac{n}{3}$, which represents all V_is apart from at most $\frac{m}{3}$ of them.

3 Fair representation by independent sets in cycles: using a theorem of Schrijver

In this section we shall prove Theorem 1.16. The proof uses a result of Schrijver [28], a strengthening of a theorem of Lovász:

Theorem 3.1 (Schrijver [28]). *For integers k, n satisfying $n > 2k$ let $K = K(n, k)$ denote the graph whose vertices are all independent sets of size k in a cycle C of length n, where two such vertices are adjacent iff the corresponding sets are disjoint. Then the chromatic number of K is $n - 2k + 2$.*

In a more straightforward formulation, this reads:

Theorem 3.2. *The family $\mathcal{I}(n, k)$ of independent sets of size k in the cycle C_n cannot be partitioned into fewer than $n - 2k + 2$ intersecting families.*

We start with a simple case, in which all V_i but one are odd:

Theorem 3.3. *Let $m, r_1, r_2, \ldots, r_m$ be positive integers, and put $n = \sum_{i=1}^{m}(2r_i + 1) - 1$. Let $G = (V, E)$ be a cycle of length n, and let $V = V_1 \cup V_2 \cup \ldots \cup V_m$ be a partition of its set of vertices into m pairwise disjoint sets, where $|V_i| = 2r_i + 1$ for all $1 \leqslant i < m$ and $|V_m| = 2r_m$. Then there is an independent set S of G satisfying $|S| = \sum_{i=1}^{m} r_i$ and $|S \cap V_i| = r_i$ for all $1 \leqslant i \leqslant m$.*

Proof of Theorem 3.3: Put $k = \sum_{i=1}^{m} r_i$ and note that $n - 2k + 2 = m + 1 > m$. Assume, for contradiction, that there is a partition with parts V_i of the set of vertices V of G as in the theorem, with no $S \in \mathcal{I}(n, k)$ satisfying the assertion of the theorem. Then for every $S \in \mathcal{I}(n, k)$ there is at least one index i for which $|S \cap V_i| \geqslant r_i + 1$. Indeed, otherwise $|S \cap V_i| \leqslant r_i$ for all i and hence $|S \cap V_i| = r_i$ for all i, contradicting the assumption. Let $\mathcal{F}_i$ be the family of sets $S \in \mathcal{I}(n, k)$ for which $|S \cap V_i| \geqslant r_i + 1$. Clearly, $\mathcal{F}_i$ is intersecting (in fact, intersecting within V_i), contradicting the conclusion of Theorem 3.2. $\square$

Corollary 3.4. *If $V = V_1 \cup V_2 \cup \cdots \cup V_m$ is a partition of the vertex set of a cycle C into m pairwise disjoint sets, then the following hold:*

(i) If $|V_i|$ is even for some index i then there is an independent set S_i of C so that $|S_i \cap V_i| = |V_i|/2$, for every j so that $|V_j|$ is odd, $|S_i \cap V_j| = (|V_j| - 1)/2$ and for every $j \neq i$ so that $|V_i|$ is even $|S \cap V_j| = |V_j|/2 - 1$.

(ii) If $|V_i|$ is odd for all i then for any vertex v of C there is an independent set S of C so that $v \notin S$ and $|S \cap V_i| = (|V_i| - 1)/2$ for all i.

Proof of Corollary 3.4: Part (i) in case all sets V_j besides V_i are of odd sizes is exactly the assertion of Theorem 3.3. If there are additional indices $j \neq i$ for which $|V_j|$ is even, choose an arbitrary vertex from each of them and contract an edge incident with it. The result follows by applying the theorem to the shorter cycle obtained. Part (ii) is proved in the same way, contracting an edge incident with v. $\Box$

4 More applications of Schrijver's theorem and its extensions

4.1 Hypergraph versions

The results above can be extended by applying known hypergraph variants of Theorem 3.1. For integers $n \geqslant s \geqslant 2$, let C_n^{s-1} denote the $(s-1)$-th power of a cycle of length n, that is, the graph obtained from a cycle of length n by connecting every two vertices whose distance in the cycle is at most $s-1$. Thus if $s = 2$ this is simply the cycle of length n whereas if $n \leqslant 2s-1$ this is a complete graph on n vertices. For integers n, k, s satisfying $n > ks$, let $K(n, k, s)$ denote the following s-uniform hypergraph. The vertices are all independent sets of size k in C_n^{s-1}, and a collection $V_1, V_2, \ldots, V_s$ of such vertices forms an edge iff the sets V_i are pairwise disjoint. Note that for $s = 2$, $K(n, k, 2)$ is exactly the graph $K(n, k)$ considered in Theorem 3.1. The following conjecture appears in [12].

Conjecture 4.1. *For $n > ks$, the chromatic number of $K(n, k, s)$ is $\lceil \frac{n-ks+s}{s-1} \rceil$.*

This is proved in [12] if s is any power of 2. Using this fact we can prove the following.

Theorem 4.2. *Let $s \geqslant 2$ be a power of 2, let m and $r_1, r_2, \ldots, r_m$ be integers, and put $n = s\sum_{i=1}^m r_i + (s-1)(m-1)$. Let $V_1, V_2, \ldots, V_m$ be a partition of the vertex set of C_n^{s-1} into m pairwise disjoint sets, where $|V_i| = sr_i + s - 1$ for all $1 \leqslant i < m$, and $|V_m| = sr_m$. Then there exists an independent set S in C_n^{s-1} satisfying $|S \cap V_i| = r_i$ for all $1 \leqslant i \leqslant m$.*

Proof: Put $k = \sum_{i=1}^m r_i$ and note that the chromatic number of $K(n, k, s)$ is $\lceil (n-ks+s)/(s-1) \rceil > m$. Assume, for contradiction, that there is a partition of the vertex set of C_n^{s-1} with parts V_i as in the theorem, with no independent set of C_n^{s-1} of size $k = \sum_{i=1}^m r_i$ satisfying the assertion of the theorem. In this case, for any such independent set S there is at least one index i so that $|S \cap V_i| \geqslant r_i + 1$. We can thus define a coloring f of the independent sets of size k of C_n^{s-1} by letting $f(S)$ be the smallest i so that $|S \cap V_i| \geqslant r_i + 1$. Since the chromatic number of $K(n, k, s)$ exceeds m, there are s pairwise disjoint sets $S_1, S_2, \ldots, S_s$ and an index i so that $|S_j \cap V_i| \geqslant r_i + 1$ for all $1 \leqslant j \leqslant s$. But this implies that $|V_i| \geqslant sr_i + s$, contradicting the assumption on the size of the set V_i, and completing the proof. $\Box$

Just as in the previous section, this implies the following.

Corollary 4.3. *Let $s > 1$ be a power of 2. Let $V_1, V_2, \ldots, V_m$ be a partition of the set of vertices of C_n^{s-1}, where $n = \sum_{i=1}^m |V_i|$, into pairwise disjoint sets. Then there is an independent set S in C_n^{s-1} so that*

$$|S \cap V_i| = \left\lfloor \frac{|V_i| - s + 1}{s} \right\rfloor$$

for all $1 \leqslant i < m$, and

$$|S \cap V_m| = \left\lfloor \frac{|V_i|}{s} \right\rfloor.$$

The proof is by contracting edges, reducing each set V_i to one of size $s\left\lfloor \frac{|V_i|-s+1}{s} \right\rfloor + s - 1$ for $1 \leqslant i < m$, and reducing V_m to a set of size $s\left\lfloor \frac{|V_m|}{s} \right\rfloor$. The result follows by applying Theorem 4.2 to this contracted graph.

4.2 The Du-Hsu-Wang conjecture

Du, Hsu and Wang [18] conjectured that if a graph on $3n$ vertices is the edge disjoint union of a Hamilton cycle of length $3n$ and n vertex disjoint triangles then its independence number is n. Erdős conjectured that in fact any such graph is 3 colorable. Using the algebraic approach in [13], Fleischner and Stiebitz [29] proved this conjecture in a stronger form - any such graph is in fact 3-choosable.

The original conjecture, in a slightly stronger form, can be derived from Theorem 3.3: omit any vertex and apply the theorem with $r_i = 1$ for all i. So, for every vertex v there exists a representing set as desired in the conjecture omitting v. The derivation of the statement of Theorem 3.3 from the result of Schrijver in [28] actually supplies a quick proof of the following:

Theorem 4.4. *Let $C_{3n} = (V, E)$ be cycle of length $3n$ and let $V = A_1 \cup A_2 \cup \ldots \cup A_n$ be a partition of its vertex set into n pairwise disjoint sets, each of size 3. Then there exist two disjoint independent sets in the cycle, each containing one point from each A_i.*

Proof. Define a coloring of the independent sets of size n in C_{3n} as follows. If S is such an independent set and there is an index i so that $|S \cap A_i| \geqslant 2$, color S by the smallest such i. Otherwise, color S by the color $n + 1$. By [28] there are two disjoint independent sets S_1, S_2 with the same color. This color cannot be any $i \leqslant n$, since if this is the case then

$$|(S_1 \cup S_2) \cap A_i| = |S_1 \cap A_i| + |S_2 \cap A_i| \geqslant 2 + 2 = 4 > 3 = |A_i|,$$

which is impossible. Thus S_1 and S_2 are both colored $n+1$, meaning that each of them contains exactly one element of each A_i. □

The Fleischner-Stiebitz theorem implies that the representing set in the HDW conjecture can be required to contain any given vertex. This can also be deduced from the topological version of Hall's Theorem proved in [6] (for this derivation see e.g [2]). The latter shows also that the cycle of length $3n$ can be replaced by the union of cycles, totalling $3n$ vertices, none being of length 1 mod 3. Simple examples show that the Fleischner-Stiebitz theorem on 3-colorability does not apply to this setting.

Note that none of the above proofs supplies an efficient algorithm for finding the desired independent set.

5 Fair representation by matchings in $K_{n,n}$, the case of two parts

The case $m = 2$ of Conjecture 1.11 is easy. Here is its statement in this case:

Theorem 5.1. *If F is a subset of $E(K_{n,n})$, then there exists a perfect matching N such that $|N \cap F| \geqslant \left\lfloor \frac{|F|}{n} \right\rfloor - 1$ and $|N \setminus F| \geqslant \left\lfloor \frac{|E(G) \setminus F|}{n} \right\rfloor - 1$.*

The fact that it is possible to reach any permutation from any other by a sequence of transpositions means that it is possible to reach every perfect matching in $K_{n,n}$ from any other by a sequence of exchanges, in each step replacing two edges of the perfect matching by two other edges. If such a sequence starts with surplus of F edges and ends with shortage of F edges, then at the stage in which the transition from surplus to shortage occurs the condition is satisfied.

Thus, in this case the interesting question is in which cases is the (-1) term necessary. That this term is sometimes necessary is shown, for example, by the case of $n = 2$ and F being a perfect matching. Another example - $n = 6$ and $F = [3] \times [3] \cup \{4, 5, 6\} \times \{4, 5, 6\}$: it is easy to see that there is no perfect matching containing precisely 3 edges from F, as required in the conjecture.

The appropriate condition is given by the following concept:

Definition 5.2. A subset F of $E(K_{n,n})$ is said to be *rigid* if there exist subsets K and L of $[n]$ such that $F = K \times L \cup ([n] \setminus K) \times ([n] \setminus L)$.

The rigidity in question is with respect to F-parity of perfect matchings:

Theorem 5.3. *[8] A subset F of $E(K_{n,n})$ is rigid if and only if $|P \cap F|$ has the same parity for all perfect matchings P in $K_{n,n}$.*

This characterization shows that when F is rigid, it is not always possible to drop the "minus 1" term in Theorem 5.1.

We shall show:

Theorem 5.4. *If a subset F of $E(K_{n,n})$ is not rigid, or if $n \nmid |F|$, then there exists a perfect matching N such that $|N \cap F| \geqslant \left\lfloor \frac{|F|}{n} \right\rfloor$ and $|N \setminus F| \geqslant \left\lfloor \frac{|E(G)\setminus F|}{n} \right\rfloor$.*

This will clearly follow from:

Theorem 5.5. *If a subset F of $E(K_{n,n})$ is not rigid then for every integer c such that $n - \nu(E(G) \setminus F) \leqslant c \leqslant \nu(F)$ there exists a perfect matching N satisfying $|N \cap F| = c$.*

This theorem, in turn, easily follows from:

Theorem 5.6. *Let $G = K_{n,n}$ and let $a < c < b$ be three integers. Suppose that $F \subseteq E(G)$ is not rigid. If there exists a perfect matching P_a such that $|P_a \cap F| = a$ and a perfect matching P_b such that $|P_b \cap F| = b$, then there exists a perfect matching P_c satisfying $|P_c \cap F| = c$.*

Proof. We use the matrix language of the original Ryser conjecture. Let M be the $n \times n$ matrix in which $m(i,j) = 1$ if $(i,j) \in F$ and $m(i,j) = 0$ if $(i,j) \notin F$. A perfect matching in G corresponds to a *generalized diagonal* (below, "g.d") in M, namely a set of n entries belonging to distinct rows and columns. A g.d will be called a *k-g.d* if exactly k of its entries are 1. By assumption there exist an a-g.d T^a and a b-g.d T^b. Assume, for contradiction, that there is no c-g.d. The case $n = 2$ is trivial, and hence, reversing the roles of 0s and 1s if necessary, we may assume that $c > 1$. Since a g.d corresponds to a permutation in S_n, and since every permutation can be obtained from any other permutation by a sequence of transpositions, there exists a sequence of g.d's $T^a = T_1, T_2, \ldots, T_k = T^b$, where each pair T_i and T_{i+1}, $i = 1, \ldots, k-1$, differ in two entries. By the negation assumption there exists i such that $T := T_{i+1}$ is a $c+1$-g.d and $T' := T_i$ is a $(c-1)$-g.d. Without loss of generality we may assume that T lies along the main diagonal and that its first $c+1$ entries are 1.

Let $I = [c+1]$, $J = [n] \setminus I$ and let $A = M[I \mid I]$, $B = M[I \mid J]$, $C = M[J \mid I]$, $D = M[J \mid J]$ (we are using here a common notation - $M[I \mid J]$ denotes the submatrix of M induced by the row set I and column set J). We may assume that the g.d T' is obtained from T by replacing the entries (c,c) and $(c+1,c+1)$ by $(c+1,c)$ and $(c,c+1)$ (Figure 1).

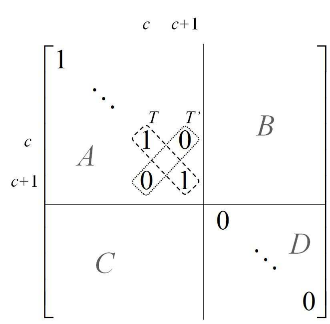

Figure 1

Claim 1. *The matrices A and D are symmetric.*

Proof of Claim 1. To prove that A is symmetric, assume, for contradiction, that there exist $i_1 \neq i_2 \in I$ such that $m_{i_1,i_2} \neq m_{i_2,i_1}$. Then, we can replace the entries (i_1, i_1) and (i_2, i_2) in T by (i_1, i_2) and (i_2, i_1) to obtain a c-g.d. The proof for D is similar, applying the replacement in this case to T'.

Claim 2. *If $i \in I$ and $j \in J$ then $m_{i,j} \neq m_{j,i}$.*

Proof of Claim 2. **Case** I**:** $m_{i,j} = m_{j,i} = 0$. Replacing (i,i) and (j,j) in T by (i,j) and (j,i) results in a c-g.d.

Case II**:** $m_{i,j} = m_{j,i} = 1$.

Subcase II_1**:** $i \notin \{c, c+1\}$. Replacing in T' the entries (i,i) and (j,j) by (i,j) and (j,i) results in a c-g.d.

Subcase II_2**:** $i \in \{c, c+1\}$. Without loss of generality we may assume $i = c+1$ and $j = c+2$ (Figure 2). If $m_{k,m} = m_{m,k} = 0$ for some $1 \leqslant k < m \leqslant c$ then replacing in T the entries $(k,k), (m,m), (c+1, c+1)$ and $(c+2, c+2)$ by $(k,m), (m,k), (c+1, c+2)$ and $(c+2, c+1)$ results in a c-g.d. (Figure 2. In all figures the removed entries are struck out by $\times$ and the added entries are circled). Thus we may assume that $m_{k,m} = m_{m,k} = 1$ for all $k, m \leqslant c$.

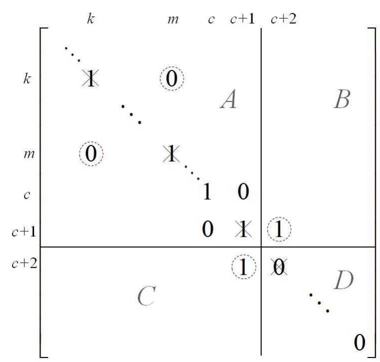

Figure 2

We now consider three sub-subcases:

(i) $m_{c,c+2} = 0, m_{c+2,c} = 1$. In this case we may replace the entries $(c,c), (c+1, c+1)$ and $(c+2, c+2)$ in T by $(c, c+2), (c+1, c)$ and $(c+2, c+1)$ and obtain a c-g.d (Figure 3(a)).

(ii) $m_{c,c+2} = 1, m_{c+2,c} = 0$. Replace the same entries as in Case (i) by $(c, c+1), (c+1, c+2)$ and $(c+2, c)$, again obtaining a c-g.d (Figure 3(b)).

(iii) $m_{c,c+2} = m_{c+2,c} = 1$. If $m_{c-1,c+1} = 0$ then, remembering that $m(c-1, c-1) = 1$, we can replace $(c-1, c-1), (c,c), (c+1, c+1)$ and $(c+2, c+2)$ in T by $(c-1, c+1), (c,c)+2), (c+1, c-1), (c+2, c)$ and obtain a c-g.d (Figure 4(a)). If $m_{c-1,c+1} = 1$, we can replace $(c-1, c-1), (c,c)$ and $(c+1, c+1)$ in T by $(c-1, c+1), (c, c-1)$ and $(c+1, c)$ and obtain a c-g.d (Figure 4(b).). This proves Claim 2.

For a matrix K indexed by any set of indices J denote by $K_{(i)}$ the ith row of K, and by $K^{(j)}$ the jth column of K.

Claim 3. *For any $j \in J$, the submatrix A is the addition table modulo 2 of the row $C_{(j)}$ and the column $B^{(j)}$. (Figure 5).*

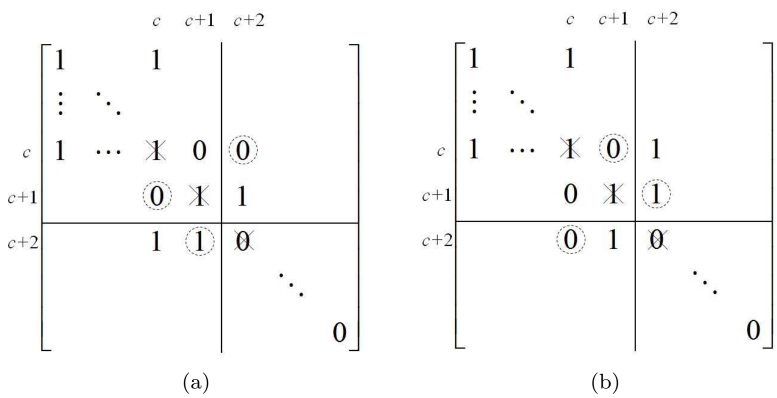

Figure 3

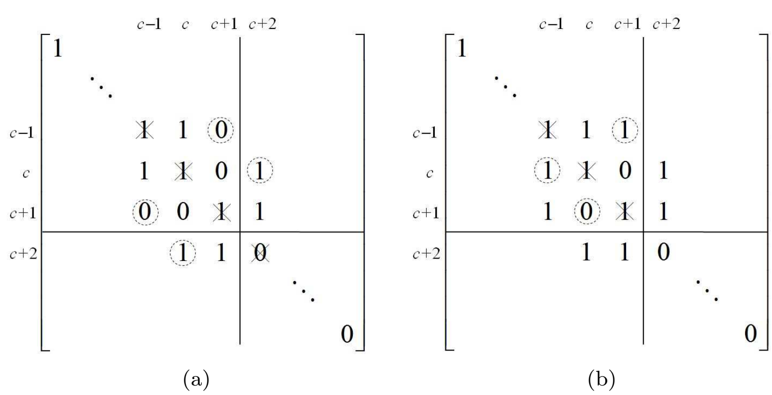

Figure 4

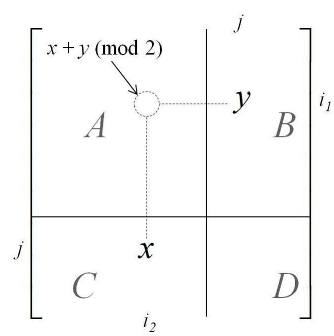

Figure 5

Proof of Claim 3. We need to show that for any $i_1, i_2 \in I$ and $j \in J$ we have $m_{i_1,i_2} = m_{j,i_2} + m_{i_1,j}$ (mod 2). We may assume that $i_1 \neq i_2$ since the case $i_1 = i_2$ follows from Claim 2 and the fact that A has 1's in the main diagonal. Let $x = m_{j,i_2} \in C_{(j)}$ and $y = m_{i_1,j} \in B^{(j)}$. We consider three cases: (i) $x \neq y$, (ii) $x = y = 0$, and (iii) $x = y = 1$.

(i) Assume, for contradiction, that $m_{i_1,i_2} = 0$. Then, by Claim 1, $m_{i_2,i_1} = 0$ and we can replace $(i_1, i_1), (i_2, i_2)$ and (j, j) in T by $(i_2, i_1), (i_1, j)$ and (j, i_2) and obtain a c-g.d (Figure 6(a)). (ii) Assume, for contradiction, that $m_{i_1,i_2} = 1$. We perform the same exchange as in Case (i) and, again, obtain a c-g.d (Figure 6(b)). (iii) By Claim 2, we have $m_{i_2,j} = m_{j,i_1} = 0$. Assume, for contradiction, that $m_{i_1,i_2} = 1$. We replace $(i_1, i_1), (i_2, i_2)$ and (j, j) in T by $(i_1, i_2), (i_2, j)$ and (j, i_1) and obtain a c-g.d (Figure 6(c)). This proves Claim 3.

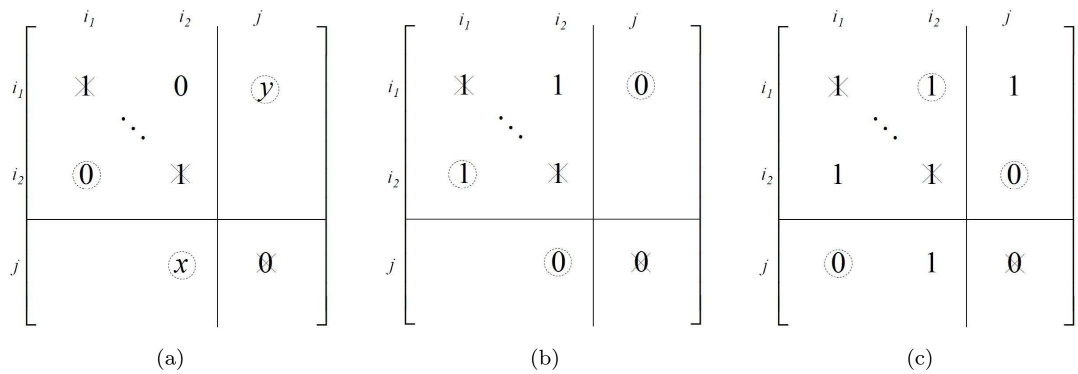

Figure 6

We say that two (0,1)-vectors u and v of the same length are *complementary* (denoted $u \bowtie v$) if their sum is the vector $(1, 1, \ldots, 1)$. By Claim 3, for every $i_1, i_2 \in I$, if for some, or equivalently any, $j \in J$, it is true that $m_{i_1,j} = m_{i_2,j}$ then the two rows $A_{(i_1)}, A_{(i_2)}$ are identical, and if $m_{i_1,j} \neq m_{i_2,j}$ then these two rows are complementary. Furthermore - the rows M_{i_1}, M_{i_2} are identical or complementary. We summarize this in:

Claim 4. *Any two rows in $M[I \mid [n]]$ are either identical or complementary.*

Next we show that the property in Claim 4 holds for any two rows in M.

For $x, y \in \{0, 1\}$ we define the operation $x \circ y = x + y + 1 \pmod 2$ (Figure 7).

Claim 5. *The submatrix D is the $\circ$-table between the column $C^{(i)}$ and the row $B_{(i)}$, for any $i \in I$.*

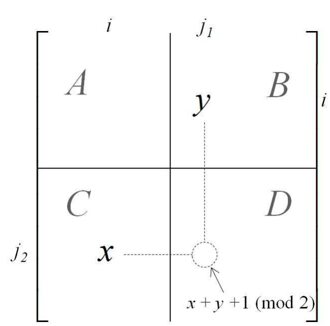

Figure 7

Proof of Claim 5. We first consider i such that $1 \leqslant i \leqslant c-1$ (we assumed $c > 1$). Let $j_1, j_2 \in J$. We may assume that $j_1 \neq j_2$ since the case $j_1 = j_2$ follows from Claim 2 and the fact that D has 0's in the diagonal. Let $x = m_{j_2,i}$ and $y = m_{i,j_1}$. We consider three cases: (i) $x = y = 0$, (ii) $x = y = 1$, and (iii) $x \neq y$.

(i) Assume, for contradiction, that $m_{j_2,j_1} = 0$. By Claim 1, $m_{j_1,j_2} = 0$, and we can replace $(i,i), (j_1,j_1)$ and (j_2,j_2) in T by $(i,j_1), (j_1,j_2)$ and (j_2,i) and obtain a c-g.d (Figure 8(a)). (ii) By Claim 2, $m_{j_1,i} = m_{i,j_2} = 0$, and we can replace the same entries as in Case 1 by (i,j_2), (j_1,i) and (j_2,j_1) and obtain a c-g.d (Figure 8(b)). (iii) Here is where we need the assumption $i \leqslant c-1$. We perform the same replacement as in Case 1, but this time on the g.d T', and obtain a c-g.d (Figure 8(c). Recall that T' is a $(c-1)$-g.d).

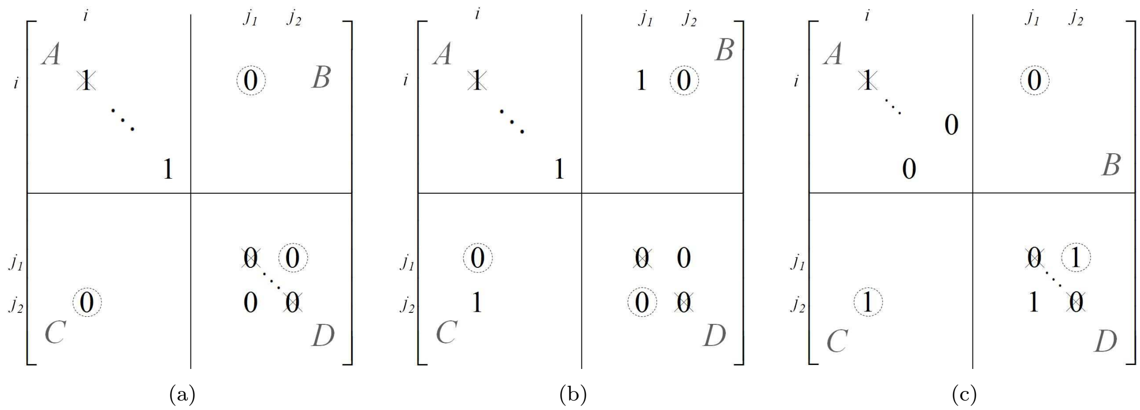

Figure 8

It remains to prove the claim for $i = c, c+1$. It follows from Claim 4 that any two rows of B are either identical or complementary. Thus, by Claim 2, any two columns of C are either identical or complementary. If there exists $j < c$ such that $B_{(c)} = B_{(j)}$, then $C^{(c)} = C^{(j)}$. Since D is the $\circ$-table between $C^{(j)}$ and $B_{(j)}$, it is also the $\circ$-table between $C^{(c)}$ and $B_{(c)}$. If all $j < c$ satisfy $B_{(c)} \bowtie B_{(j)}$, then for any such j, we have $C^{(c)} = B^T_{(j)}$ and $C^{(j)} = B^T_{(c)}$ by Claim 2. Since $\circ$ is commutative we again have that D is the $\circ$-table between $C^{(c)}$ and $B_{(c)}$. A similar argument holds for $i = c+1$.

Claim 6. *Any two rows of M are either identical or complementary.*

Proof of Claim 6. The fact that any two rows in $M[J|[n]]$ are either identical or complementary follows in the same manner as Claim 4. Now, assume $i \in I, j \in J$. We want to show that $M_{(i)}$ is either identical or complementary to $M_{(j)}$. From Claim 3 we know that $A_{(i)}$ is either identical or complementary to $C_{(j)}$ and from Claim 5 we have that $B_{(i)}$ is either identical or complementary to $D_{(j)}$. We need to show that $A_{(i)}$ is identical to $C_{(j)}$ if and only if $B_{(i)}$ is identical to $D_{(j)}$. Note that $m_{ii} = 1$, $m_{jj} = 0$ and $m_{ij} \neq m_{ji}$. So, if $m_{ji} = 1$ we have identity in both cases and if $m_{ji} = 0$ we have complementarity in both cases.

Suppose all the rows of M are identical. Then, the first $c+1$ columns are all-1 and the rest of the columns are all-0. So, any g.d has exactly $c+1$ 1s. So, $a = b = c+1$, which is obviously not the case. Thus, by Claim 6, we can permute the rows and columns to obtain a matrix M' consists of four submatrices M_1, M_2, M_3 and M_4 of positive dimensions, where M_1 and M_4 are all-1, and M_2 and M_3 are all-0 (Figure 9).

Thus, F is rigid (Definition 5.2), contrary to the hypothesis. We conclude that there must be a c-g.d in M. □

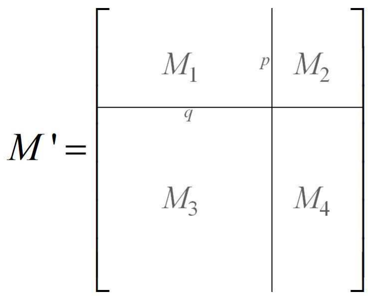

Figure 9

In the case that the partition $E(G) = F \cup (E(G) \setminus F)$ is rigid, if there exists a partition P_{c+1} such that $|P_{c+1} \cap F| = c+1$, then clearly there is no partition P_c such that $|P_c \cap F| = c$. The proof of Theorem 5.6 shows that in this case, for any c between a and b there is a partition $P_{c'}$ such that $0 \leqslant |P_{c'} \cap F| - c \leqslant 1$.

Corollary 5.7. *Let $G = K_{n,n}$ and assume the partition $E(G) = F \cup (E(G) \setminus F)$ is not rigid. Then, there exist perfect matchings P_1 and P_2 such that $|P_1 \cap F| = \left\lfloor \frac{|F|}{n} \right\rfloor$ and $|P_2 \cap F| = \left\lceil \frac{|F|}{n} \right\rceil$.*

6 Fair representation by perfect matchings in $K_{n,n}$, the case of three parts

6.1 Statement of the theorem and outline of proof

In this section we prove Conjecture 1.11 for $m = 3$, namely:

Theorem 6.1. *Suppose that the edges of $K_{n,n}$ are partitioned into sets E_1, E_2, E_3. Then, there exists a perfect matching F in $K_{n,n}$ satisfying $\left\lceil \frac{|E_i|}{n} \right\rceil + 1 \geqslant |F \cap E_i| \geqslant \left\lfloor \frac{|E_i|}{n} \right\rfloor - 1$ for every $i = 1, 2, 3$.*

We identify perfect matchings in $K_{n,n}$ with permutations in S_n. For $\sigma, \tau \in S_n$, the *Hamming distance* (or plainly *distance*) $d(\sigma, \tau)$ is $|\{i \mid \sigma(i) \neq \tau(i)\}|$. We write $\sigma \sim \tau$ if $d(\sigma, \tau) \leqslant 3$. Let $\mathcal{C}$ be the simplicial complex of the cliques of this relation.

The proof will show, in effect, that $\eta(\mathcal{C}) \geqslant 3$. This will enable us to use Sperner's lemma. Here is a short outline of this proof. Clearly, for each $i \leqslant 3$ there exits a matching F_i representing E_i fairly, namely $|F_i \cap E_i| \geqslant \left\lfloor \frac{|E_i|}{n} \right\rfloor$. We shall connect every pair F_i, F_j by a path consisting of perfect matchings representing fairly $E_i \cup E_j$, in which every two adjacent matchings are $\sim$-related. This forms a triangle D, which is not necessarily simple, namely it may have repeating vertices, a triangulation T of its circumference, and an assignment A of matchings to its vertices. We shall then show that there exists a triangulation T' extending T and contained in $\mathcal{C}$. Here "contained in $\mathcal{C}$" means that there is an assignment A' of perfect matchings to the vertices of T', that extends A, such that the perfect matchings assigned to adjacent vertices are $\sim$-related. We color a vertex v of T' by color i if $A'(v)$ represents $\frac{1}{n}$-fairly the set E_i. By our construction, this coloring satisfies the conditions of Sperner's lemma, and applying the lemma we obtain a multicolored triangle. We shall then show that at least one of the matchings assigned to the vertices of this triangle satisfies the condition required in the theorem.

6.2 A hexagonal version of Sperner's Lemma

In order to prove theorem 6.1 we shall use the following version of Sperner's lemma.

Lemma 6.2. *Let T be a piecewise linear image of the disc B^2, and assume that its boundary is a (not necessarily simple) hexagon H, with edges $e_1, \ldots, e_6$. Suppose that the vertices of H are colored $1, 2, 3$. Assume that*

- *No vertex in the triangulation of e_1 has color 1.*
- *No edge in the triangulation of e_2 is between two vertices of colors 1 and 2.*
- *No vertex in the triangulation of e_3 has color 2.*
- *No edge in the triangulation of e_4 is between two vertices of colors 2 and 3.*
- *No vertex in the triangulation of e_5 has color 3.*
- *No edge in the triangulation of e_6 is between two vertices of colors 3 and 1.*

Then T contains a triangle between three vertices colored 1, 2 and 3.

Proof. We add three vertices to T utside the circumference of the hexagon so that a complex with a triangular boundary is formed: a vertex of color 1 adjacent to all vertices in the triangulation of e_4, a vertex of color 2 adjacent to all vertices in the triangulation of e_6 and a vertex of color 3 adjacent to all vertices in the triangulation of e_2. We can then use the original Sperner's Lemma on this augmented complex. □

6.3 Moving between permutations

Let $i \in [n] := \{1, \ldots, n\}$. We define a function $shift_i : S_n \to S_n$ as follows: For every $\sigma \in S_n$, if $\sigma(i) = j$ then for every $k \in [n]$ we define

$$shift_i(\sigma)(k) = \begin{cases} i & \text{if } k = i \\ j & \text{if } \sigma(k) = i \\ \sigma(k) & \text{otherwise} \end{cases}$$

Remark 6.3. Note that if $\sigma(i) = i$ then $shift_i(\sigma) = \sigma$.

The aim of this operation is to eliminate i in its cycles expression, meaning obtaining a permutation sending i to itself. The application of $shift_i$ results in this state: if i is followed by j in some cycle of σ, then i is removed from this cycle.

Lemma 6.4. *Let n be a positive number, Let $i \in [n]$ and let $\sigma, \tau \in S_n$. If $\sigma \sim \tau$ then $shift_i(\sigma) \sim shift_i(\tau)$.*

Proof. For simplicity we write $shift$ for $shift_i$. Without loss of generality $i = 1$. If $shift(\sigma) = \sigma$ and $shift(\tau) = \tau$ then we are done.

The second case we need to consider is that $shift(\tau) = \tau$ but $shift(\sigma) \neq \sigma$. Without loss of generality $\tau = I$. For every $k \in [n]$, if $\sigma(k) = k$ then also $shift(\sigma)(k) = k$ and thus the distance between $shift(\sigma)$ and I is at most the distance between σ and I, yielding $shift(\sigma) \sim I = shift(\tau)$.

We are left with the case where $shift(\sigma) \neq \sigma$ and $shift(\tau) \neq \tau$. Without loss of generality $\tau = (12)$ and hence $shift(\tau) = I$. As in the previous case, for every $k \in [n]$ if $\sigma(k) = k$ then also $shift(\sigma)(k) = k$. We also note that $shift(\sigma)(1) = 1$ but $\sigma(1) \neq 1$ (since $shift(\sigma) \neq \sigma$). Therefore, the distance between $shift(\sigma)$ and I is strictly less than the distance between σ and I. If the distance between σ and I is at most 4 then $shift(\sigma) \sim I = shift(\tau)$ and we are done. Since $\sigma \sim \tau$, this distance cannot be more than 5, so it must be exactly 5. Note that if $\sigma(1) = j \neq 2$, then σ and τ differ on 1,2 and j, and thus $\sigma(k) = k$ for all $k \notin \{1, 2, j\}$, so the distance between σ and I is at most 3, contrary to the assumption that this distance is 5. Thus, we must have that $\sigma(1) = 2$. It follows that the set $A = \{i \in [n] : \sigma(i) \neq \tau(i)\}$ is a set of size 3 disjoint from $\{1, 2\}$. But then also $\{i \in [n] : shift(\sigma(i)) \neq shift(\tau(i))\} = A$, yielding $shift(\sigma) \sim shift(\tau)$. □

6.4 Path connectivity

Lemma 6.5. *Let $A = (a_{ij})$ be an $n \times n$ 0-1 matrix and let $k \in [n-1]$. Let G be the graph whose vertices are the permutations $\sigma \in S_n$ satisfying $\sum_{i=1}^n a_{i\sigma(i)} \geqslant k$ and whose edges correspond to the $\sim$ relation. If there exists $\rho \in S_n$ with $\sum_{i=1}^n a_{i\rho(i)} > k$, then G is connected.*

Proof. Without loss of generality ρ is the identity permutation, namely $\sum_{i=1}^n a_{ii} > k$. We shall show that there is a path in G from ρ to σ for any $\sigma \in V(G) \setminus \{\rho\}$. We prove this claim by induction on $d(\sigma, \rho)$. Write $\ell = \sum_{i=1}^n a_{i\sigma(i)}$. Our aim is to find distinct $j \in [n]$ for which $\sigma(j) \neq j$ and $\sigma' = shift_j(\sigma) \in V(G)$. Then the induction hypothesis can be applied since $\sigma \sim \sigma'$ and σ' is closer to ρ than σ.

If $\ell \geqslant k+2$ choose any $j \in [n]$ with $\sigma(j) \neq j$. Then we have $\sum_{i=1}^n a_{i\sigma'(i)} \geqslant \sum_{i=1}^n a_{i\sigma(i)} - 2 \geqslant k$, so $\sigma' \in V(G)$.

Suppose next that $\ell = k+1$. By the assumption that $\sum_{i=1}^n a_{ii} > k$ we have $\sum_{i=1}^n a_{i\sigma(i)} \leqslant \sum_{i=1}^n a_{ii}$ and since $\sigma \neq \rho$ there must be some $j \in [n]$ for which $\sigma(j) \neq j$ and $a_{jj} \geqslant a_{j\sigma(j)}$. Taking $\sigma' = shift_j(\sigma) \in V(G)$ yields $\sum_{i=1}^n a_{i\sigma'(i)} \geqslant \sum_{i=1}^n a_{i\sigma(i)} - 1 = k$, so $\sigma' \in V(G)$.

Finally, if $\ell = k$ then $\sum_{i=1}^n a_{i\sigma(i)} < \sum_{i=1}^n a_{ii}$ and hence there must be some $j \in [n]$ for which $a_{jj} > a_{j\sigma(j)}$. Taking $\sigma' = shift_j(\sigma) \in V(G)$ we get $\sum_{i=1}^n a_{i\sigma'(i)} \geqslant \sum_{i=1}^n a_{i\sigma(i)} + 1 - 1 = k$, so $\sigma' \in V(G)$. □

Corollary 6.6. *Let $A = (a_{ij})$ be an $n \times n$ 0-1 matrix and let $k \in [n]$. Let G be the graph whose vertices are the permutations $\sigma \in S_n$ with $\sum_{i=1}^n a_{i\sigma(i)} \geqslant k$ and whose edges correspond to the $\sim$ relation. If $\sum_{i,j \leqslant n} a_{ij} \geqslant kn$ then G is connected.*

Proof. If there exists a permutation ρ with $\sum_{i=1}^n a_{i\rho(i)} > k$ then we are done by Lemma 6.5. If not, by König's theorem there exist sets $A, B \subseteq [n]$ with $|A| + |B| \leqslant k$ such that $a_{ij} = 0$ for $i \notin A$ and $j \notin B$. This is compatible with the condition $\sum_{i,j \leqslant n} a_{ij} \geqslant kn$ only if $|A| = 0$ and $|B| = k$ or $|B| = 0$ and $|A| = k$, and $a_{ij} = 1$ for all $(i, j) \in A \times [n] \cup [n] \times B$. In both cases $V(G) = S_n$, implying that the relation $\sim$ is path connected since every permutation is reachable from every other permutation by a sequence of transpositions. □

6.5 Reaching a super-fair and under-fair representation simultaneously

Lemma 6.7. *Let the set E of edges of $K_{n,n}$ be partitioned to three sets $E = E_1 \dot\cup E_2 \dot\cup E_3$. Then there exists a perfect matching M with at least $\lceil \frac{|E_1|}{n} \rceil$ edges of E_1 and at most $\lceil \frac{|E_3|}{n} \rceil$ edges of E_3.*

Proof. Let H be the graph with the edge set $E_1 \cup E_2$. König's edge coloring theorem, combined with an easy alternating paths argument, yields that H can be edge colored with n colors in a way that each color class is of size either $\left\lfloor \frac{|E(H)|}{n} \right\rfloor$ or $\left\lceil \frac{|E(H)|}{n} \right\rceil$. Clearly, at least one of these classes contains at least $\frac{|E_1|}{n}$ edges from E_1. A matching with the desired property can be obtained by completing this color class in any way we please to a perfect matching of $K_{n,n}$. □

We conjecture that a stronger property holds:

Conjecture 6.8. *Let $G = (V, E)$ be a bipartite graph with maximal degree Δ and let $f : E \to \{1, 2, 3, \dots, k\}$ for some positive integer k. Then there exists a matching M in G such that every number $j \in \{1, 2, 3, \dots, k\}$ satisfies*

$$|\{e \in M \mid f(e) \leqslant j\}| \geqslant \left\lfloor \frac{|\{e \in E : f(e) \leqslant j\}|}{\Delta} \right\rfloor$$

Clearly, we only need to see to it that the condition holds for $j < k$.

In [15], this conjecture was proved for $G = K_{6,6}$.

6.6 A Sperner coloring of the boundary of a hexagon

From this point until the end of this section, we assume that the set E of edges of $K_{n,n}$ is partitioned to three sets $E = E_1 \dot\cup E_2 \dot\cup E_3$ and for each $i = 1, 2, 3$ we have $|E_i| = k_i n$ where k_i is integer. We assume that Theorem 6.1 does not hold, so in particular, there is no perfect matching with exactly k_i edges from each E_i. As mentioned in the previous section, Theorem 6.1 is easy in the case one of the sets E_i is empty, so we may assume $k_1, k_2, k_3 \in \{1, \dots, n-2\}$.

A perfect matching is said to have property $i+$ if it has at least k_i edges from E_i, it is said to have property $i++$ if it has strictly more than k_i edges from E_i and it is said to have property $i-$ if it has at most k_i edges from E_i

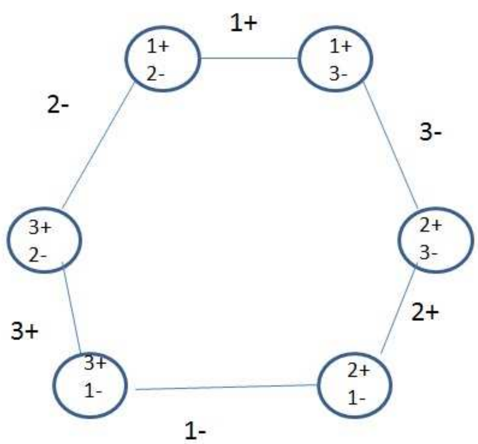

Figure 10

Lemma 6.9. *There exists a triangulation of the boundary of a hexagon, and an assignment of a perfect matching M_v and a color $i_v \in \{1, 2, 3\}$ to each vertex v of the triangulation, such that M_v has property i_v++ and the coloring satisfies the conditions of Lemma 6.2.*

Proof. By Lemma 6.7 there exists a perfect matching M with properties $1+$ and $3-$. We assign it to one vertex of the hexagon. By permuting the roles of E_1, E_2, E_3 we can find six such perfect matchings and assign them to the six vertices of the hexagon as in Figure 10.

By Corollary 6.6, we can fill the path between the two permutations with property $i-$ in a way that all perfect matchings in the path have property $i-$. Similarly, we can fill the path between the two permutations with property $i+$. For each vertex v we assign a color i_v such that M_v has property i_v++. Now suppose Lemma 6.2 does not hold, then without loss of generality we have two perfect matchings

$M_1 \sim M_2$, where M_1 has properties $3+$ and $1++$ and M_2 has properties $3+$ and $2++$ and this easily gives Theorem 6.1. □

6.7 Simple connectivity

In the next two lemmas let n be a positive integer, $i \in [n]$ and $\sigma.\tau \in S_n$. We write $shift$ for $shif_i$.

Lemma 6.10. *If $d(\sigma, \tau) = 2$, then the 4-cycle $\sigma - \tau - shift(\tau) - shift(\sigma) - \sigma$ is a null cycle in $\mathcal{C}$ (i.e., it can be triangulated.)*

Proof. If either $\sigma \sim shift(\tau)$ or $\tau \sim shift(\sigma)$ then we are done. So, we may assume this does not happen and in particular $\sigma \neq shift(\sigma)$ and $\tau \neq shift(\tau)$. We may assume, without loss of generality, that $i = 1$, $\sigma = (12)$, $\tau = (12)(34)$, $shift(\sigma) = I$ and $shift(\tau) = (34)$. We can now fill the cycle as in Figure 11. □

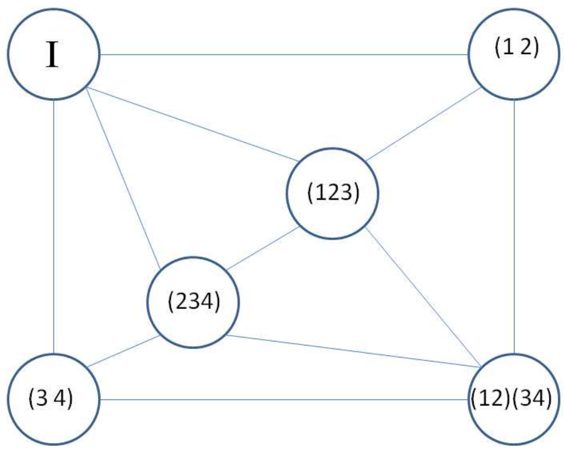

Figure 11

Lemma 6.11. *If $d(\sigma, \tau) = 3$ then the 4-cycle $\sigma - \tau - shift(\tau) - shift(\sigma) - \sigma$ is a null cycle in $\mathcal{C}$.*

Proof. Let $\rho \in S_n$ have distance 2 from both σ and τ. Denote $\sigma' = shift(\sigma)$, $\tau' = shift(\tau)$ and $\rho' = shift(\rho)$. We use the previous lemma to fill the cycle as in Figure 12. □

Lemma 6.12. *The simplicial complex $\mathcal{C}$ is simply connected.*

Proof. Let $f : S^1 \to ||\mathcal{C}||$ be a continuous function and let $Z = Imf$. We need to show that Z is a null cycle in $\mathcal{C}$ by finding a continuous function $g : B^2 \to ||\mathcal{C}||$ extending f. We arbitrarily pick distinct $i \in [n]$, and whenever a point $p \in S^1$ has $f(p) = \sigma \in Z$, we define $g(p/2) = \sigma' = shift_i(\sigma)$. We now use the previous two lemmas to fill the outer shell of B^2. (See Figure 13.)

We repeat this for all i and eventually, we are left with a cycle all of whose points are sent to the identity permutation. We define g to be I in the interior of this cycle as well and we are done. □

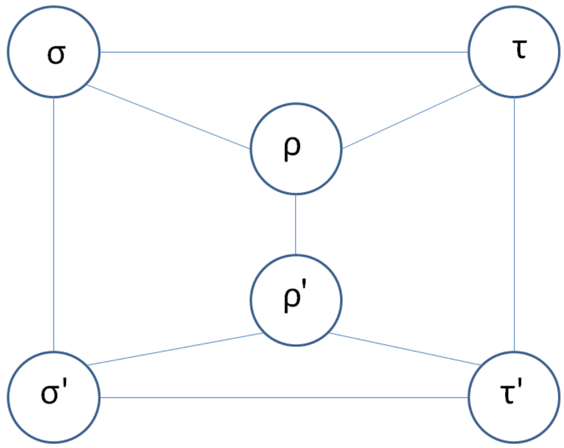

Figure 12

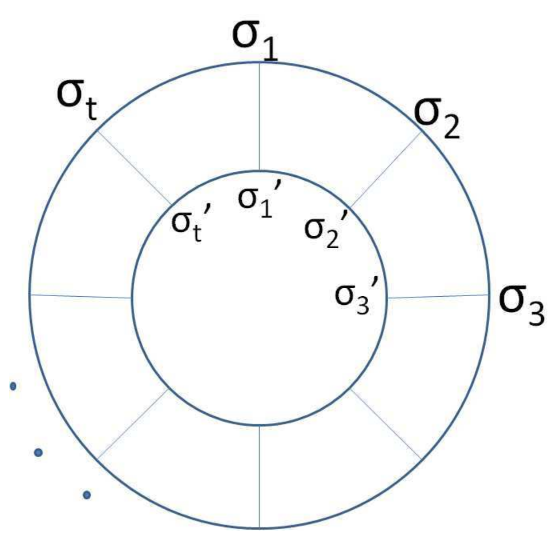

Figure 13

6.8 Applying Sperner's lemma

We form a matrix $A = (a_{ij})_{i,j \leqslant n}$, where $a_{ij} = p\ \ (p = 1, 2, 3)$ if the edge ij belongs to E_p.

For each $\ell \in \{1, 2, 3\}$ and $\sigma \in S_n$ we write $d_\ell(\sigma) = |\{i : a_{i\sigma(i)} = \ell\}| - k_\ell$.

Our next lemma points at what we may aspire for, in order to prove the theorem.

Lemma 6.13. *Suppose that the triple $\{\sigma_1, \sigma_2, \sigma_3\}$ is in $\mathcal{C}$, and that $d_\ell(\sigma_\ell) > 0$ for each $\ell \in \{1, 2, 3\}$. Then there exists $\sigma \in S_n$ with $|d_\ell(\sigma)| \leqslant 1$ for each $\ell \in \{1, 2, 3\}$.*

Since the existence of such $\sigma_1, \sigma_2, \sigma_3$ follows from Lemmas 6.2, 6.9 and 6.12, this will finish the proof of Theorem 6.1.

Proof. Define a 3×3 matrix $B = (b_{ij})$ by $b_{ij} = d_i(\sigma_j)$. We know that the diagonal entries in B are positive, the sum in each column is zero, and any two entries in the same row differ by at most 3. This means that the minimal possible entry in B is -2. We may assume each column has some entry not in $\{-1, 0, 1\}$.

Let us start with the case that all of the diagonal entries of B are at least 2. This implies that all off-diagonal entries are at least -1. Since each column must sum up to zero, we must have

$$B = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

This implie that the distance between any two of $\sigma_1, \sigma_2, \sigma_3$ is exactly 3, and without loss of generality $\sigma_1 = I$, $\sigma_2 = (123)$, $\sigma_3 = (132)$, and the matrix A has the form

$$A = \begin{pmatrix} 1 & 2 & 3 & * & \dots & * \\ 3 & 1 & 2 & * & \dots & * \\ 2 & 3 & 1 & * & \dots & * \\ * & * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & \dots & * \end{pmatrix}$$

We can now take $\sigma = (12)$ and we are done.

We are left with the case that some diagonal entry of B is 1. Without loss of generality $b_{11} = 1$. We also assume without loss of generality that $b_{21} \leqslant b_{31}$. Since the first column must sum up to zero, we have $b_{21} + b_{31} = -1$, and thus $-0.5 = 0.5(b_{21} + b_{31}) \leqslant b_{31} = -1 - b_{21} \leqslant 1$. In other words, either $b_{21} = -1$ and $b_{31} = 0$ or $b_{21} = -2$ and $b_{31} = 1$. In the first case we can just take $\sigma = \sigma_1$ and we are done. Therefore we assume the second case.

$$B = \begin{pmatrix} 1 & * & * \\ -2 & * & * \\ 1 & * & * \end{pmatrix}$$

Since $d_3(\sigma_1) > 0$, we may assume $\sigma_3 = \sigma_1$, and due to the -2 entries in the second row, we must have $b_{22} = 1$. We now get

$$B = \begin{pmatrix} 1 & * & 1 \\ -2 & 1 & -2 \\ 1 & * & 1 \end{pmatrix}$$

Without loss of generality $b_{12} \leqslant b_{32}$ and by arguments similar to the above we can fill the second column

$$B = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}$$

The distance between σ_1 and σ_2 is exactly 3, so without loss of generality $\sigma_1 = I$ and $\sigma_2 = (123)$. In order to achive the values of $b_{12} = -2, b_{11} = 1, b_{21} = -2, b_{22} = 1$ we must have $a_{ii} = 1$ and $a_{i\sigma_2(i)} = 2$ for each $i \in \{1, 2, 3\}$.

The only case in which none of the choices $\sigma = (12)$ or $\sigma = (23)$ or $\sigma = (13)$ works is if $a_{13} = a_{21} = a_{32} = 3$, so once again we get

$$A = \begin{pmatrix} 1 & 2 & 3 & * & \dots & * \\ 3 & 1 & 2 & * & \dots & * \\ 2 & 3 & 1 & * & \dots & * \\ * & * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & \dots & * \end{pmatrix}$$

We have $b_{31} = 1$ which means that 3 appears $k_3 + 1$ times on the diagonal. Without loss of generality $a_{44} = a_{55} = \ldots = a_{k_3+4\,k_3+4} = 3$. In any of the following cases one can easily find some $\sigma \in S_n$ with $|d_\ell(\sigma)| \leqslant 1$ for each $\ell \in \{1, 2, 3\}$:

- If either $a_{ij} \neq 3$ or $a_{ji} \neq 3$ for some $i \in \{4, \ldots, k_3 + 4\}$ and $j \in \{1, 2, 3\}$.
- If $a_{ij} \neq 3$ for some $i, j \in \{4, \ldots, k_3 + 4\}$
- If both $a_{ij} \neq 3$ and $a_{ji} \neq 3$ for some $i \in \{4, \ldots, k_3 + 4\}$ and $j \in \{k_3 + 5, \ldots, n\}$.

If none of the above occures then

$$k_3 n = |\{(i, j) : a_{ij} = 3\}| \geqslant 2 \cdot 3 \cdot (1 + k_3) + (1 + k_3)^2 + \frac{1}{2} \cdot 2(k_3 + 1)(n - k_3 - 4)$$

which is a contradiction. □

Remark 6.14. After the articulation of the above topological proof of Theorem 6.1, a combinatorial proof was given in [16].

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