# On the Hat Guessing Number of Graphs

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**Abstract**

The hat guessing number $HG(G)$ of a graph $G$ on $n$ vertices is defined in terms of the following game: $n$ players are placed on the $n$ vertices of $G$, each wearing a hat whose color is arbitrarily chosen from a set of $q$ possible colors. Each player can see the hat colors of his neighbors, but not his own hat color. All of the players are asked to guess their own hat colors simultaneously, according to a predetermined guessing strategy and the hat colors they see, where no communication between them is allowed. The hat guessing number $HG(G)$ is the largest integer $q$ such that there exists a guessing strategy guaranteeing at least one correct guess for any hat assignment of $q$ possible colors.

In this note we construct a planar graph $G$ satisfying $HG(G) = 12$, settling a problem raised in [4]. We also improve the known lower bound of $(2 - o(1)) \log_2 n$ for the typical hat guessing number of the random graph $G = G(n, 1/2)$, showing that it is at least $n^{1-o(1)}$ with probability tending to 1 as $n$ tends to infinity. Finally, we consider the linear hat guessing number of complete multipartite graphs.

## 1 Introduction

The following hat guessing game was introduced in [5]. Let $G$ be a simple graph on $n$ vertices $\{v_1, \ldots, v_n\}$, and let $Q$ be a finite set of $q$ colors. The $n$ vertices of the graph are identified with $n$ players, where each is assigned arbitrarily a hat colored with one of the colors in $Q$. A player can only see the hat colors of his neighbors, i.e., player $i$ sees the hat color of player $j$ if and only if $v_i$ is connected to $v_j$ in $G$. After all the players agree on a guessing strategy, they are asked to guess their own hat colors simultaneously, with no communication. The

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goal of the players is to ensure that at least one player guesses his hat color correctly. Let $HG(G)$ denote the maximum number $q$ for which there exists a winning guessing strategy for the players.

If $q$ is a prime power, the set $Q$ is identified with the set of elements of the finite field $F = GF(q)$, and the guessing functions are linear (or affine), we call the guessing strategy a linear strategy.

The invariant $HG(G)$ has been studied in several papers including [5, 8, 11, 15, 7, 1, 4, 10, 9]. In [4] the authors conjecture that the maximum possible hat guessing number of a planar graph is 4. The following result shows that it is significantly larger.

**Theorem 1.1.** *There exists a planar graph with hat guessing number 12.*

We have established this result in September 2020 (as mentioned in [9]). At that time this has been the planar graph with the largest hat guessing number known. Examples of planar graphs with hat guessing number 6 appear in [10] and in [9]. A more recent paper [12] contains a construction of a planar graph with hat guessing number 14.

Another conjecture suggested in [4] is that the hat guessing number of any graph is at most its Hadwiger's number, that is, the order of the largest clique minor of the graph. The theorem above provides, of course, a counterexample to that as well. For larger Hadwiger numbers $d$ the hat guessing number can in fact be at least doubly-exponential in $d$. This follows from the results in [10], as it is easy to show that the Hadwiger number of the graphs $G_d(N)$ constructed there and discussed in Theorem 1.1 in that paper is $d+1$. Similarly (though less dramatically) the book graph $B_{d,m}$ for $m > m_0(d)$ has Hadwiger number $d+1$ and hat guessing number exceeding $d^d$.

It is interesting to determine or estimate the hat guessing number of the random graph $G = G(n, 1/2)$. In [4] it is shown that with high probability, that is, with probability tending to 1 as $n$ tends to infinity, $(2 - o(1)) \log_2 n \leq HG(G) \leq n - (1 + o(1)) \log_2 n$. The following result improves the lower bound considerably.

**Theorem 1.2.** *Let $G = G(n, 1/2)$ denote the binomial random graph. Then with high probability, that is, with probability tending to* 1 *as $n$ tends to infinity, $HG(G) \geq n^{1-o(1)}$.*

Our final result in this note deals with linear guessing strategies. Here $q$ is a prime power, the colors are identified with the elements of the finite field $F = GF(q)$, and each vertex guesses according to an affine function of the colors of its neighbors.

**Definition 1.3** ([1]). *The linear hat guessing number of $G$, denoted $HG_{lin}(G)$, is the largest prime power $q$ for which there is a winning linear guessing strategy for $G$ over the field $GF(q)$*

The linear hat guessing number of $K_{n,n}$, the complete bipartite graph with $n$ vertices in each color class, is studied in [1]. Here we consider the complete $m$-partite graph, $K_n^{(m)}$, with $n$ vertices in each color class.

**Theorem 1.4.** *There exists an absolute constant $c$ so that the following holds. Let $F$ be a field with at least $(c\log(nm))^m$ elements. Then there is no linear hat guessing scheme for $K_n^{(m)}$ over $F$.*

The above results are proved in the next three sections. The final brief section contains several open problems.

# 2 Proof of Theorem 1.1

*Proof of Theorem 1.1.* Let $G$ be the planar graph consisting of two adjacent vertices $u, v$ together with $m = 66^{144}$ pairs of adjacent vertices $x_i, y_i$, each adjacent to both $u$ and $v$. See Figure 1, and note that we make no attempt to minimize $m$, which can be easily reduced. Call $uv$ the central pair of $G$, and all edges $x_iy_i$ the outer pairs. Let $\mathcal{F}$ denote the family of all $m = 66^{144}$ functions $f$ from ordered pairs of elements of the cyclic group $\mathbf{Z}_{12}$ to unordered pairs of elements of $\mathbf{Z}_{12}$. Associate each outer pair with one such function according to some fixed bijection between these functions and the outer pairs. The guessing strategies of the vertices in the outer pairs are defined as follows. Consider such a pair $xy$, let $f$ be the function corresponding to it, and let $h(u), h(v), h(x), h(y)$ be the hat colors of $u, v, x$ and $y$, respectively, viewed as elements of $\mathbf{Z}_{12}$. The value of $f(h(u), h(v))$ is an unordered pair of elements $\{g_1, g_2\}$ of $\mathbf{Z}_{12}$ with, say, $g_1 < g_2$. Let $x$ guess that its hat color is $g_1 - h(y)$ (computed in $\mathbf{Z}_{12}$) and let $y$ guess that its hat color is $g_2 - h(x)$ (computed in $\mathbf{Z}_{12}$.) Note that both $x, y$ guess incorrectly if and only if the sum of hat colors of $x, y$ is not one of the two elements in $f(h(u), h(v))$. We claim that for any fixed hat colors of all vertices in the outer pairs there are at most 5 distinct colorings of the hats of $u, v$ in which all vertices in the outer pairs guess incorrectly. Indeed, if there are at least 6 such colorings then there is some function $f \in \mathcal{F}$ (in fact many such functions) that maps these 6 colorings to pairwise disjoint pairs of elements of $\mathbf{Z}_{12}$ whose union covers all of $\mathbf{Z}_{12}$. Let $x, y$ be an outer pair corresponding to this function. Then, since $x, y$ fail to guess correctly in all these 6 colorings, it follows that the sum of the hat colors of $x, y$ cannot be any element of $\mathbf{Z}_{12}$, a contradiction. This proves the claim and shows that in case all the vertices in the outer pairs guess incorrectly there are only 5 possible distinct colorings of the pair $u, v$. The vertices $u, v$ see all hats of the vertices in the outer pairs and hence can compute this set of at most 5 distinct colorings of their hats. They can then take care of these colorings ensuring that at least one of them guesses correctly since as shown in [4, 10] a clique of size 2 can handle any (known) set of 5 colorings. This shows that the hat guessing number of our graph is at least 12. We next show that it is smaller than 13 (and hence it is exactly 12).

Consider the game on this graph with 13 colors. We will pre-commit to using a member of a fixed set of 6 colorings for the vertices of the central pair. For each coloring of these vertices, each vertex in an outer pair can guess correctly in at most 13 possible colorings of the outer pair, one for each color of the other vertex in the pair. Hence each outer pair can handle at most 26 different colorings of the pair. Therefore for all 6 colorings of the central pair, this outer pair can have at least one vertex guessing correctly in at most $26 \cdot 6 = 156$ possible colorings of the pair. Since there are $13^2 = 169$ possible colorings of an outer pair, there is at least one (in fact, at least $169 - 156 = 13$) coloring where they guess incorrectly for all 6 colorings of the central pair. For each outer pair we can choose a coloring in which both its vertices guess incorrectly for all 6 colorings. This shows that for any fixed set of 6 possible colorings of the central pair, there are choices for the colors of all outer pairs (for any number of outer pairs) in which all vertices of the outer pairs guess incorrectly.

As shown in [4, 10] there exists a set of 6 colorings, for example the set $\{0,1\} \times \{0,1,2\}$, such that the central vertices cannot guess correctly on all of them. Suppose the central pair gets one of these 6 colorings, and every outer pair gets a coloring in which both its members guess incorrectly for any of these 6 colorings. Each of the two vertices in the central pair sees the colors of all other vertices, and knows the guessing function of each vertex. Crucially, even with this information, each of these two vertices knows that the central pair may well have any of the above 6 colorings. As for any strategy of the central pair, both its members guess incorrectly on at least one of these 6 colorings, this shows that there is no winning strategy for the graph with 13 colors. Therefore the hat guessing number of this graph is exactly 12. $\square$

## 3 Proof of Theorem 1.2

Our lower bound on the hat guessing number of the random graph uses the fact that the random graph contains a a sufficiently large book graph as a subgraph with high probability.

The book graph denoted $B_{d,m}$ is the graph obtained by taking a complete graph on $d$ vertices, which we call the central clique, adding to it an independent set of size $m$, and joining each of its vertices to all the vertices of the central clique. Figure 2 shows a drawing of $B_{3,9}$. The hat guessing numbers of book graphs are studied in [4], [10], [9]. The following result is proved in [9].

**Theorem 3.1** ([9]). *For every fixed $d$ and $m$ sufficiently large as a function of $d$,*

$$HG(B_{d,m}) = \sum_{i=1}^{d} i^i + 1.$$

We cannot use this result to prove Theorem 1.2, since the assertion above holds only for $m$ which is very large as a function of $d$. For our purpose we need to show that the hat

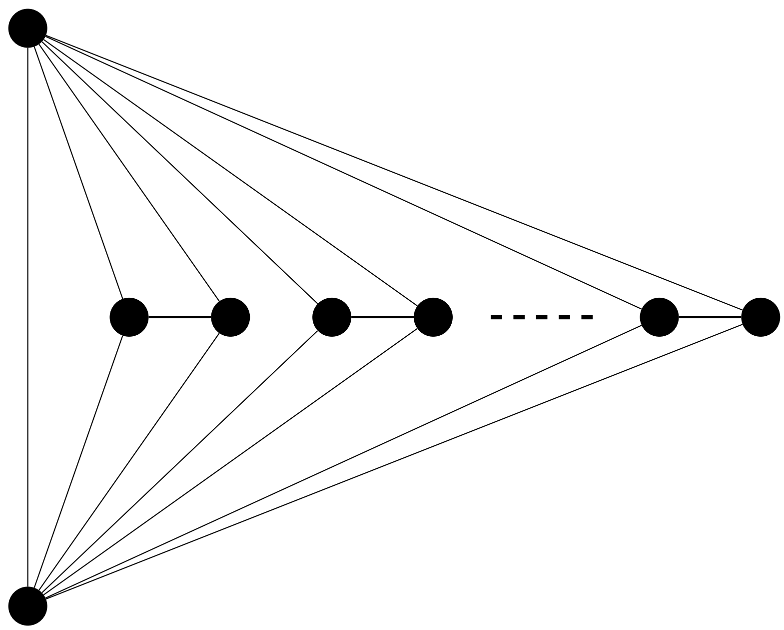

Figure 1: A planar graph with hat guessing number 12.

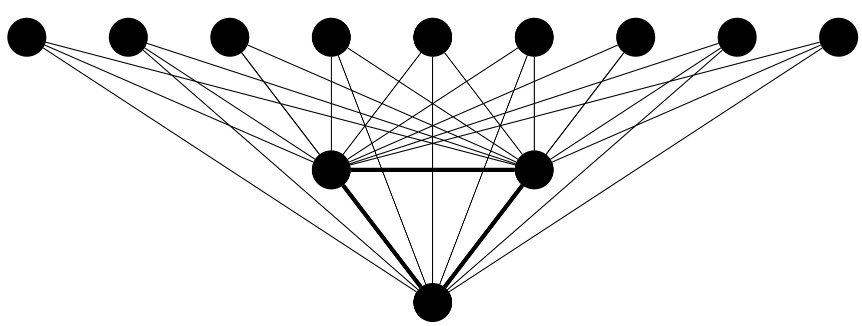

Figure 2: The book graph $B_{3,9}$

guessing number of $B_{d,m}$ is close to the expression above even if $m$ is much smaller. This is proved in the following lemma. It is worth noting that both the value of $m$ and that of $q$ can be slightly improved in this lemma, but the estimate below suffices for our purpose.

**Lemma 3.2.** *For $m = d^d \cdot d^3, HG(B_{d,m}) \geq q = d^d/d^2$.*

*Proof.* Let $q = d^d/d^2 = d^{d-2}$. Call the clique of $B$ the central clique, and the other vertices the outer ones. We claim that there are $m$ functions $f_1, \dots, f_m$ from $\mathbf{Z}_q^d$, the set of possible colorings of the central clique to $\mathbf{Z}_q$ so that for every subset $S \subset \mathbf{Z}_q^d$ of size $|S| = d^d$ at least one of the functions restricted to $S$, $f_i|_S$ is onto. Indeed, choose these functions randomly and uniformly among all possible functions from $\mathbf{Z}_q^d$ to $\mathbf{Z}_q$. For a fixed set, $S$, the probability that a fixed function misses a specific element of $\mathbf{Z}_q$ is $(1-1/q)^{d^d}$. By the union bound, the probability that at least one element is missed is at most

$$q(1-1/q)^{d^d} \leq qe^{-d^d/q} < qe^{-d^2} < 1/2$$

Thus the probability that none of the functions is onto is less than $(1/2)^m = (1/2)^{d^d \cdot d^3}$. The number of subsets of size $d^d$ of the set of colorings is

$$\binom{q^d}{d^d} < d^{d^d \cdot d^2} = 2^{d^d \cdot d^2 \cdot \log d}$$

and the claim follows from the union bound as $2^{d^d \cdot d^2 \cdot \log d} \cdot (1/2)^{d^d \cdot d^3} < 1$, so with positive probability, for every subset $S \subset \mathbf{Z}_q^d$ of size $|S| = d^d$ there is a function $f_i|_S$ which is onto $\mathbf{Z}_q$, proving the existence of a set of $m$ such functions. Returning to the proof of the lemma let $f_1, \dots, f_m$ be as in the claim, and let these be the guessing functions of the outer vertices of $B$. By the claim it is clear that for any fixed hat colors of the outer vertices there are less than $d^d$ colorings of the central clique for which all outer vertices guess incorrectly, since for every set of $d^d$ colorings at least one of the guessing functions is onto. Thus the central vertices can compute this known set of less than $d^d$ colorings and as shown in [4, 10, 9] can then handle them. $\square$

*Proof of Theorem 1.2.* It is well known (c.f., e.g., [2]) that with high probability, the graph $G = G(n, 1/2)$ contains a clique of size $(2 + o(1)) \log_2 n$. Let $d$ be the largest number so that $d^d \cdot d^3 \leq 0.5n/2^d$. This gives $d = (1 + o(1)) \log n / \log \log n$. Let $K$ be a clique of size $d$ in $G$. Such a clique exists with high probability since $d$ is (much) smaller than $2 \log_2 n$. The expected number of its common neighbors is $(n-d)/2^d$ and by the standard estimates for binomial distributions (c.f., e.g., [2], Appendix A), with high probability it has more than $0.5n/2^d \geq d^d \cdot d^3$ common neighbors. This means that with high probability $G$ contains a book $B_{d,m}$ with $m = d^d \cdot d^3$ and the result follows from Lemma 3.2. $\square$

# 4 Proof of Theorem 1.4

The result is proved using an idea in [3], where the authors obtain an improved bound for the sunflower lemma, and its modifications in [6, 14, 16].

**Definition 4.1** ([16], Definition 1). *We call a family, $\mathcal{F}$, of sets, $R$-spread if for every nonempty set $Z$ a uniformly random set $F$ chosen from $\mathcal{F}$ satisfies $\mathbb{P}(S \subseteq F) \leq R^{-|Z|}$*

**Lemma 4.2** (Proposition 5 of [16] (following [3, 6, 14])). *Let $r, w \geq 2$ be natural numbers, and let $R \geq Cr \log(wr)$ for a sufficiently large absolute constant $C$ not dependent on $r, w$. Let $\mathcal{F} = (F_i)_{i \in I}$ be a finite, $R$-spread family of subsets of a finite set $T$, each of which having cardinality at most $w$. Let $V$ be a random subset of $T$ where each $t \in T$ independently lies in $V$ with probability $1/r$. Then with probability greater than $1 - 1/r$, $V$ contains an element of $\mathcal{F}$.*

*Proof of Theorem 1.4.* Denote the vertices of $K_n^{(m)}$ by $v_{ij}$, $1 \leq i \leq n, 1 \leq j \leq m$, let $x_{ij} \in F$ be the hat-value of $v_{ij}$, put $p = |F|$ and let the linear guessing function of $v_{ik}$ be

$$f_{ik}(x_{ij} : 1 \leq i \leq n, 1 \leq j \leq m, j \neq k) + b_{ik}$$

Put $w = nm$, let $T := [n] \times [m] \times Z_p$, with $|T| = pnm$. For each of the $p^{nm}$ possible vectors $x = (x_{ik}) \in Z_p^{nm}$ define two subsets of size $w = nm$ of $T$ as follows.

$$F(x) = \left\{ \Big(i, k, x_{ik} - f_{ik}(x_{ij} : 1 \leq i \leq n, 1 \leq j \leq m, j \neq k) - b_{ik}\Big) : 1 \leq i \leq n, 1 \leq k \leq m \right\},$$

$$G(x) = \left\{ \Big(i, k, x_{ik} - f_{ik}(x_{ij} : 1 \leq i \leq n, 1 \leq j \leq m, j \neq k)\Big) : 1 \leq i \leq n, 1 \leq k \leq m \right\}$$

Let $\mathcal{F}$ be the family of all $p^{nm}$ sets $F(x)$ as above, and let $\mathcal{G}$ be the family of all $p^{nm}$ sets $G(x)$. It is easy to check that each of these two families is $R$-spread for $R = p^{1/m}$. Indeed, any set containing some given fixed set $Z$ must satisfy some specific family of $|Z|$ linear equations. At least $|Z|/m$ of these equations correspond to the same value of $k$, by the pigeonhole principle, and these equations are linearly independent, as each $x_{ik}$ that appears among them appears only in one of them. So the number of solutions is at most $p^{w-|Z|/m}$, and the total number of sets is $p^w$. By Lemma 4.2 with $r = 2$ there are sets $F \in \mathcal{F}$ and $G \in \mathcal{G}$ which are disjoint. Indeed, if we color the elements of $T$ randomly, uniformly and independently by 2 colors then with positive probability the first color class contains a member of $\mathcal{F}$ and the second a member of $\mathcal{G}$.

Taking the difference provides a vector of colors $x = (x_{ij})$ for which none of the differences $x_{ik} - (f_{ik}(x) + b_{ik})$ vanishes. This means that for the corresponding coloring no vertex guesses correctly, completing the proof. $\square$

# 5 Open Problems

One of the questions raised in [1] is whether there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if $G$ is $d$-degenerate, then $HG(G) \leq f(d)$. It seems plausible that such a function exists. If so, then the maximum possible hat guessing number of any planar graph is bounded by an absolute constant, as planar graphs are 5 degenerate. More generally, this would imply that the hat guessing number of any graph can be upper bounded by a function of its Hadwiger number, as graphs with Hadwiger number $k$ are known to be $O(k\sqrt{\log k})$-degenerate, see [13], [17].

Another interesting question is the typical asymptotic behavior of the hat guessing number of the random graph $G(n, 1/2)$. In particular, is it $o(n)$ with high probability?

The final question we suggest here is that of estimating the largest possible hat guessing number of a graph $G'$ obtained from a graph $G$ with hat guessing number $q$ by the addition of a single vertex (connected to all vertices of $G$). The graphs $G_d(N)$ constructed in [10] can be used to show that for arbitrarily large values of $q$ the number can increase to more than $q^2$. This is obtained by letting $G$ be the (disconnected) graph obtained from $G' = G_d(N)$ by deleting its unique vertex which is connected to all other ones. Thus every connected component of $G$ is a copy of $G_{d-1}(N)$. As shown in [10], for sufficiently large $N$, if the hat guessing number of $G$ (which equals that of each of its connected components) is $q$, then the hat guessing number of $G'$ is $q^2 + q$.

It will be interesting to establish an upper bound for $HG(G')$ as a function of $HG(G)$, or to show that no such function exists.

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