Bipartite Subgraphs of Integer Weighted Graphs

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Abstract

For every integer $p > 0$ let $f(p)$ be the minimum possible value of the maximum weight of a cut in an integer weighted graph with total weight p. It is shown that for every large n and every $m < n$, $f(\binom{n}{2} + m) = \lfloor \frac{n^2}{4} \rfloor + \min(\lceil \frac{n}{2} \rceil, f(m))$. This supplies the precise value of $f(p)$ for many values of p including, e.g, all $p = \binom{n}{2} + \binom{m}{2}$ when n is large enough and $\frac{m^2}{4} \leq \frac{n}{2}$.

1 Introduction

All graphs in this paper contain no loops. For a simple graph $G = (V, E)$, let $f(G)$ denote the maximum number of edges in a bipartite subgraph of G. For every $p > 0$, let $g(p)$ denote the minimum value of $f(G)$, as G ranges over all simple graphs with p edges. Thus, $g(p)$ is the largest integer such that any simple graph with p edges contains a bipartite subgraph with at least $g(p)$ edges. There are several papers providing bounds for $g(p)$, see, e.g, [1]-[5]. Edwards [2],[3] proved that for every p

$$g(p) \geq \lceil \frac{p}{2} + \frac{-1 + \sqrt{8 \cdot p + 1}}{8} \rceil.$$

Equality holds for all complete graphs. Alon [1] showed that there is a positive constant C so that

$$g(p) \leq \frac{p}{2} + \sqrt{\frac{p}{8}} + C \cdot p^{1/4},$$

for all p. He also proved that there is a positive constant c and infinitely many positive integers p satisfying

$$g(p) \geq \frac{p}{2} + \sqrt{\frac{p}{8}} + c \cdot p^{1/4}.$$

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We raise the following conjecture:

Conjecture 1.1 *For every n and for every $0 \le m < n$, $g(\binom{n}{2} + m) = \lfloor \frac{n^2}{4} \rfloor + \min(\lceil \frac{n}{2} \rceil, g(m))$.*

If Conjecture 1.1 is true then given p, we can construct an extremal graph $G = (V, E)$ with p edges for which $f(G) = g(p)$ as follows. Let n_1 be the greatest integer such that $\binom{n_1}{2} \le p$ and let $p_1 = p - \binom{n_1}{2}$. For every $i > 1$ define, by induction, n_i to be the greatest integer such that $\binom{n_i}{2} \le p_{i-1}$ if $p_{i-1} > 0$ and $n_i = 0$ otherwise and let $p_i = p_{i-1} - \binom{n_i}{2}$. Let k be an integer such that $\lfloor \frac{n_1^2}{4} \rfloor + \lfloor \frac{n_2^2}{4} \rfloor + \ldots + \lfloor \frac{n_{k-1}^2}{4} \rfloor + \lfloor \frac{n_k^2}{4} \rfloor + \lceil \frac{n_k}{2} \rceil$ is minimal. If Conjecture 1.1 is true then $g(p) = \lfloor \frac{n_1^2}{4} \rfloor + \lfloor \frac{n_2^2}{4} \rfloor + \ldots + \lfloor \frac{n_{k-1}^2}{4} \rfloor + \lfloor \frac{n_k^2}{4} \rfloor + \lceil \frac{n_k}{2} \rceil$. For every $1 \le i < k$ let G_i be the complete graph on n_i vertices. Let G_k be an arbitrary simple graph on $n_k + 1$ vertices with p_{k-1} edges. If Conjecture 1.1 is true, then an extremal graph G is obtained by taking the disjoint union of $G_1, G_2, \ldots, G_k$.

An *integer weighted graph* is a graph $G = (V, E)$ in which each edge has a (not necessary positive) integral weight. For an integer weighted graph $G = (V, E)$, let $f(G)$ denote the maximum total weight in a bipartite subgraph of G. For every $p > 0$, let $f(p)$ denote the minimum value of $f(G)$, as G ranges over all integer weighted graphs with total weight p. Thus, $f(p)$ is the largest integer such that any integer weighted graph with total weight p contains a bipartite subgraph with total weight no less than $f(p)$. The following conjecture is stronger than Conjecture 1.1:

Conjecture 1.2 *for every n and for every $0 \le m < n$ $f(\binom{n}{2} + m) = \lfloor \frac{n^2}{4} \rfloor + \min(\lceil \frac{n}{2} \rceil, f(m))$.*

If Conjecture 1.2 is true then so is Conjecture 1.1 and this implies that $f(p) = g(p)$ for every $p > 0$. Therefore, we consider the function f here and prove the following theorem:

Theorem 1.3 *Let $G = (V, E)$ be an integer weighted graph with total weight $\binom{n}{2} + m$, where $0 \le m < n$. Then $f(G) \ge \lfloor \frac{n^2}{4} \rfloor + \min(\lceil \frac{n}{2} \rceil, f(m))$, provided n is sufficiently large. Therefore, $f(\binom{n}{2} + m) = \lfloor \frac{n^2}{4} \rfloor + \min(\lceil \frac{n}{2} \rceil, f(m))$.*

In Section 2 we give an outline of the proof of Theorem 1.3, and Section 3 describes the complete proof of the theorem.

2 An Outline of the Proof

It is obvious that $f(\binom{n}{2} + m) \le \lfloor \frac{n^2}{4} \rfloor + \min(\lceil \frac{n}{2} \rceil, f(m))$ because we can construct integer weighted graphs implying the inequality. Thus it suffices to prove the first part of the theorem, i.e, to prove that given an integer weighted graph $G = (V, E)$ with weight function w and with total weight $\binom{n}{2} + m$, $f(G) \ge \lfloor \frac{n^2}{4} \rfloor + \min(\lceil \frac{n}{2} \rceil, f(m))$. We assume that $f(G) < \lfloor \frac{n^2}{4} \rfloor + \min(\lceil \frac{n}{2} \rceil, f(m))$ and get a contradiction.

To do so, we first observe that nonpositive weighted edges can be contracted, and therefore we can assume that $|V| \leq n$ and that for every $u, v \in V$, $w(u,v) > 0$. Next we show in Corollary 3.2 and Lemma 3.3 that there is a small set $U \subseteq V$ such that for every $u, v \in V \setminus U$ we have $w(u,v) = 1$.

We view the graph G as a complete graph with another graph on top of it (the multiple edges), where every vertex of U represents a set of vertices in this virtual graph which are forced to be in the same side in every bipartite graph that we take, while every vertex of $V \setminus U$ represents one vertex in the virtual graph. If $v_i \in U$ and the average weight of the edges (v, v_i) where $v \in V \setminus U$ is d_i, then v_i represents a set of d_i vertices in this virtual graph.

We next show (see Lemma 3.6) that in this way the number of represented vertices is at least n, since otherwise the total weight of the edges is less then $\binom{n}{2} + m$, contradicting the assumption.

Next, it is shown that there are two possible cases. The first is that most of the vertices behave as we view them, i.e for most of the vertices of $V \setminus U$ the weight of the edges between them and a vertex of U is the average weight of the edges between that vertex and $V \setminus U$. In this case, it is shown in Corollary 3.9 that the number of represented vertices is at most n and thus is exactly n. We take the graph which is 'on top' of the complete graph and find a large weighted bipartite subgraph in this graph and then we add the vertices which are not in this graph and get, using the induction hypothesis, a bipartite subgraph with sufficiently large total weight.

In the second case (see Lemma 3.7 and Lemma 3.8) it is shown that $w(U, V \setminus U)$ is very large. In this case we take the $\lfloor \frac{n}{2} \rfloor$ vertices of $V \setminus U$ connected to U by the heaviest edges and put them in one side and all the other vertices in the other side. As a result of the fact that the total weight $w(U, V \setminus U)$ of edges between U and $V \setminus U$ is very large, we get a bipartite subgraph of sufficiently large weight.

3 The proof of the main Theorem

We start with some definitions. Let $G = (V, E)$ be an integer weighted graph with weight function $w : E \mapsto Z$. Let r be defined by the following equation:

$$f(m) = \frac{m}{2} + \frac{r}{4}.$$

It is easy and known (see [1]) that $g(m) \leq \frac{m}{2} + \sqrt{\frac{m}{8}} + C \cdot m^{\frac{1}{4}}$ where C is an absolute constant. Since $f(m) \leq g(m)$, $\frac{r}{4} \leq \sqrt{\frac{m}{8}} + C \cdot m^{\frac{1}{4}}$. For every two sets of vertices U, W let $w(U, W)$ denote the total weight of the edges with one end in U and the other end in W, i.e $w(U,W) = \sum_{u \in U, w \in W} w(u,v)$. If U or W are singletons we write the vertex they contain instead of the corresponding sets. Define a *multimatching* M of G to be a partition of the vertices of G into pairs except for maybe one vertex if $|V|$ is odd. If $v, u \in V$ and $(v, u) \in M$ we say that v is *matched* with u by M. The *value* of the multimatching M is defined to be :

$$|M| = \sum_{(v,u)\in M} (w(v, u) - 1)$$

A *maximum multimatching* is a multimatching of maximum value.

It is obvious that

$$f(\binom{n}{2} + m) \leq \lfloor \frac{n^2}{4} \rfloor + \min(\lceil \frac{n}{2} \rceil, f(m))$$

because we can actually construct integer weighted graphs implying the inequality. Therefore it suffices to prove that given an integer weighted graph $G = (V, E)$ with weight function w and with total weight $\binom{n}{2} + m$, $f(G) \geq \lfloor \frac{n^2}{4} \rfloor + \min(\lceil \frac{n}{2} \rceil, f(m))$.

Assume G has total weight $\binom{n}{2} + m$ and $f(G) < \lfloor \frac{n^2}{4} \rfloor + \min(\lceil \frac{n}{2} \rceil, f(m))$. We assume that for every $v, u \in V$ $w(v, u) > 0$, since otherwise we could define a new graph G' by contracting the vertices v, u into one vertex v, thus getting that the total weight of G' is not less than the total weight of G. We can reduce weight from G' to get a graph G'' with total weight equal to that of G and with $f(G'') \leq f(G)$. Thus, we assume without loss of generality that G is a multigraph such that any two vertices of G are connected by at least one edge. In such a multigraph with p edges it is known (see [2]) that $f(G) \geq \frac{p}{2} + \sqrt{\frac{p}{8}} - O(1)$. Therefore, $\frac{r}{4} \geq \sqrt{\frac{m}{8}} - O(1)$ and thus $\frac{r^2}{2}$ is $m \cdot (1 + o(1))$.

Suppose then that $V = \{v_1, v_2, \ldots, v_{n-k}\}$ where $k \geq 0$ by the above observations. We need a lemma about multimatchings:

Lemma 3.1 *Let $H = (V, E)$ be a multigraph with $w(u, v) > 0$ for every $u, v \in V$ and let M be a maximum multimatching in H. If $|V|$ is even then $|E| \leq (|V| - 1) \cdot (|M| + \frac{|V|}{2})$. If $|V|$ is odd then $|E| \leq |V| \cdot (|M| + \frac{(|V|-1)}{2})$.*

Proof: First, assume that $|V|$ is even. Let $(u_1, w_1), \ldots, (u_l, w_l)$ be the pairs in the multimatching M, where $l = \frac{|V|}{2}$. for every $i \neq j$ we have

$$w(\{u_i, w_i\}, \{u_j, w_j\}) - 4 \leq 2 \cdot (w(u_i, w_i) + w(u_j, w_j) - 2)$$

because of the maximality of $|M|$. So, if we sum up over all $1 \leq i \leq j \leq 2 \cdot t$ we have $|E| \leq (2 \cdot (l - 1) + 1) \cdot (|M| + l) = (|V| - 1) \cdot (|M| + \frac{|V|}{2})$. If $|V|$ is odd we can add a vertex u to V and add an edge (u, v) for every $v \in V$. So, by the first part of the lemma, the number of edges in the new graph is bounded by $(|V|) \cdot (|M| + \frac{|V|+1}{2})$, so $|E| \leq |V| \cdot (|M| + \frac{|V|+1}{2}) - |V| = |V| \cdot (|M| + \frac{|V|-1}{2})$
□

Let M be a maximum multimatching in G. We next prove two simple bounds on the value of M.

Corollary 3.2 $|M| \geq k$.

Proof: By Lemma 3.1 we have:

$$|M| + \frac{n-k}{2} \geq \frac{|E|}{n-k} = \frac{m + \binom{k}{2} + \binom{n-k}{2} + k\cdot(n-k)}{n-k} \geq \frac{n-k}{2} + k - \frac{1}{2}$$

Thus, $|M| \geq k$. □

Lemma 3.3 $|M| \leq \frac{k+r}{2}$.

Proof: Assume $|M| > \frac{k+r}{2}$. If we take the pairs of M in a greedy way we get:

$$f(G) \geq \frac{|E| + |M|}{2} + \frac{n-k-1}{4} = \frac{n^2}{4} + \frac{m}{2} - \frac{k+1}{4} + \frac{|M|}{2} \geq \frac{n^2}{4} + f(m)$$

□

By Corollary 3.2 and Lemma 3.3 we have the following corollary:

Corollary 3.4 $k \leq r$.

Let $a := 2\cdot|M| - k$, then $|M| = \frac{k+a}{2}$, and by Lemma 3.3 and Corollary 3.2 $k \leq a \leq r$. Let $U = \{v \in V | \exists u\ ,\ (v,u) \in M\ and\ w(v,u) > 1\}$. Let $2t$ denote the size of U. We have $2t \leq 2|M| = k+a$. We assume without loss of generality that $U = \{v_1, \ldots, v_{2t}\}$. For every $1 \leq i \leq 2\cdot t$ define h_i and d_i to be the integers such that $w(v_i, V \setminus U) = d_i\cdot|V \setminus U| + h_i$ where $|h_i| \leq \frac{|V\setminus U|}{2}$ (in case $|h_i| = \frac{|V\setminus U|}{2}$ let h_i be positive). Furthermore, let x_i denote the number of vertices $v \in V \setminus U$ such that $w(v, v_i) \neq d_i$.

Lemma 3.5 *Let $v_i \in U$. Let v_j be the matched vertex of v_i in M. Then $max(|h_i|, x_i) \leq \frac{r-a}{2} + w(v_i, v_j)$.*

Proof: Without loss of generality, $w(v_i, v_{2t+1}) \geq w(v_i, v_{2t+2}) \geq \ldots \geq w(v_i, v_{n-k})$. Let $p := \lfloor\frac{r-a}{2}\rfloor + w(v_i, v_j)$. We show that $max(|h_i|, x_i) \leq p$. Let $A_1 = \{v_{2t+1}, \ldots, v_{2t+p+1}\}$. Let $A_2 = \{v_{n-k-p}, \ldots, v_{n-k}\}$. Let $B_1 = A_1$ and $B_2 = A_2 \cup \{v_i\}$ and add v_j to B_1 or to B_2 with equal probability. Now, the expected value of $w(B_1, B_2)$ is:

$$w(A_1, A_2) + w(v_i, A_1) + \frac{w(v_j, \{v_i\} \cup A_1 \cup A_2)}{2}$$

As $A_1, A_2 \subseteq V \setminus U$, the subgraph of G induced on $A_1 \cup A_2$ is a complete graph on $2\cdot p + 2$ vertices. Thus, $w(A_1, A_2) = \frac{w(A_1 \cup A_2)}{2} + \frac{|A_1|}{2}$. Furthermore, $w(v_i, A_1) = \frac{w(v_i, A_1 \cup A_2)}{2} + \frac{w(v_i, A_1) - w(v_i, A_2)}{2}$ and $w(A_1 \cup A_2 \cup \{v_i, v_j\}) = w(A_1 \cup A_2) + w(v_i, A_1 \cup A_2) + w(v_j, \{v_i\} \cup A_1 \cup A_2)$. Therefore, the expected value of $w(B_1, B_2)$ is:

$$\begin{aligned} & \frac{w(A_1 \cup A_2)}{2} + \frac{A_1}{2} + \frac{w(v_i, A_1 \cup A_2)}{2} + \frac{w(v_i, A_1) - w(v_i, A_2)}{2} + \frac{w(v_j, \{v_i\} \cup A_1 \cup A_2)}{2} \\ = & \frac{w(A_1 \cup A_2 \cup \{v_i, v_j\}) + |A_1| + (w(v_i, A_1) - w(v_i, A_2))}{2} \end{aligned}$$

If $w(v_{2t+p+1}, v_i) > w(v_{n-k-p}, v_i)$ then $w(v_i, A_1) - w(v_i, A_2) \geq p + 1$. Otherwise it is easy to see that $w(v_i, A_1) - w(v_i, A_2) \geq max(|h_i|, x_i)$. Now, if the lemma was false then we have $w(v_i, A_1) - w(v_i, A_2) \geq p + 1$ in both cases. If we continue to join in a greedy way the other pairs of M trying to maximize the weight in the cut we get a bipartite graph of size at least

$$\frac{|E|}{2} + \frac{|A_1|}{2} + \frac{w(v_i, A_1) - w(v_i, A_2)}{2} + \frac{n - k - |A_1| - |A_2| - 2 - 1}{4} + \frac{|M| - (w(v_i, v_j) - 1)}{2} \geq$$

$$\geq \frac{n^2}{4} + \frac{m}{2} - \frac{1}{4} + \frac{p+1}{2} + \frac{a}{4} - \frac{w(v_i, v_j)}{2} \geq$$

$$\geq \frac{n^2}{4} + \frac{m}{2} + \frac{r}{4},$$

contradiction. □

Lemma 3.6 $\sum_{i=1}^{2t}(d_i - 1) \geq k$.

Proof: Assume that $\sum_{i=1}^{2t}(d_i - 1) \leq k - 1$. By Lemma 3.1 we have

$$w(U) \leq (2\cdot t - 1)\cdot(|M| + t) \tag{1}$$

also,

$$w(V \setminus U) = \binom{n - k - 2\cdot t}{2} \tag{2}$$

By the definition of d_i and h_i we know that $w(V \setminus U, U) = \sum_{i=1}^{2t}(d_i\cdot|V \setminus U| + h_i)$. Now,

$$\begin{aligned} & w(V \setminus U, U) = \\ = \;& \sum_{i=1}^{2t}((d_i - 1)\cdot|V \setminus U| + h_i) + 2\cdot t\cdot|V \setminus U| \\ \leq \;& |V \setminus U|\cdot\sum_{i=1}^{2t}(d_i - 1) + \sum_{i=1}^{2t}|h_i| + 2\cdot t\cdot|V \setminus U| \\ \leq \;& (k - 1)\cdot(n - k - 2\cdot t) + \sum_{i=1}^{2t}|h_i| + 2\cdot t\cdot(n - k - 2\cdot t) \end{aligned}$$

It is easy to see that by Lemma 3.5 we have

$$\sum_{i=1}^{2t}|h_i| \leq 2\cdot|M| + 2\cdot t + (r - a)\cdot t$$

By the last inequality and by equations (1) and (2) we get that

$$\begin{aligned} & |E| \leq \\ \leq \;& (2\cdot t - 1)\cdot(|M| + t) + \binom{n - k - 2\cdot t}{2} + (k - 1)\cdot(n - k - 2\cdot t) + \end{aligned}$$

$$
\begin{aligned}
+ \quad & 2\cdot|M| + 2\cdot t + (r-a)\cdot t + 2\cdot t\cdot(n-k-2\cdot t) \\
= \quad & (\binom{n-k-2\cdot t}{2} + 2\cdot t\cdot(n-k-2\cdot t) + \binom{2\cdot t}{2}) + \\
+ \quad & (k-1)(n-k) + |M| + 2\cdot t\cdot(|M| - k + 2 + \frac{r-a}{2}) \\
= \quad & \binom{n-k}{2} + (k-1)\cdot(n-k) + |M| + 2\cdot t\cdot(\frac{r-a}{2} + 2 - k + |M|) \\
= \quad & \binom{n-k}{2} + (k-1)\cdot(n-k) + \frac{k+a}{2} + 2\cdot t\cdot(\frac{r-k}{2} + 2) \\
\leq \quad & \binom{n-k}{2} + (k-1)\cdot(n-k) + \frac{k+r}{2} + (k+r)\cdot(\frac{r-k}{2} + 2) \\
\leq \quad & |E| - m - \binom{k}{2} - n + \frac{7}{2}\cdot k + \frac{5}{2}\cdot r + \frac{r^2-k^2}{2}
\end{aligned}
$$

If $k = 2$ then the last expression attains its maximum value which is

$$
|E| - m - n + 4 + \frac{5}{2}\cdot r + \frac{r^2}{2} < |E|
$$

The last inequality follows from the fact that n is large enough, $m < n$ and $\frac{r^2}{2} = m\cdot(1 + o(1))$. $\square$

Lemma 3.7 $k \leq \frac{a}{\sqrt{2}}$.

Proof: Assume the opposite, i.e $k > \frac{a}{\sqrt{2}}$. By Lemma 3.1 we have $w(U) \leq (2\cdot t - 1)\cdot(|M| + t) = t\cdot(k+a) - \frac{k+a}{2} + \binom{2t}{2}$. Also, we have $w(V \setminus U) = \binom{n-k-2t}{2}$. Hence,

$$
\begin{aligned}
& w(U, V \setminus U) = \\
= \quad & |E| - w(V \setminus U) - w(U) \\
\geq \quad & |E| - \binom{n-k-2\cdot t}{2} - (t\cdot(k+a) - \frac{k+a}{2} + \binom{2\cdot t}{2}) \\
= \quad & (n-k-2\cdot t)(2\cdot t + k) + 2\cdot t\cdot k + \binom{k}{2} + m - t\cdot(k+a) + \frac{k+a}{2} \\
= \quad & (n-k-2\cdot t)(2\cdot t + k) + t\cdot(k-a) + \binom{k}{2} + m + \frac{k+a}{2}
\end{aligned}
$$

As $a \geq k$ and $t \leq \frac{k+a}{2}$ we get that $t\cdot(k-a) \geq \frac{k^2-a^2}{2}$.Combining with the last inequality we get:

$$
w(U, V \setminus U) \geq (n-k-2\cdot t)(2\cdot t + k) + m + k^2 - \frac{a^2}{2}
$$

Now, if $k > \frac{a}{\sqrt{2}}$ then it follows that

$$
w(U, V \setminus U) \geq (n-k-2\cdot t)\cdot(2\cdot t + k) + m \tag{3}
$$

Without loss of generality, $U = \{v_1, \ldots, v_{2t}\}$ and $w(v_{2t+1}, U) \geq w(v_{2t+2}, U) \geq \ldots \geq w(v_{n-k}, U)$. Let $A = \{v_{2t+1}, \ldots, v_{2t+\lceil \frac{n}{2} \rceil}\}$. If $w(v_{2t+\lceil \frac{n}{2} \rceil}, U) \geq 2{\cdot}t + k + 1$ we get that $w(v_{2t+i}, V \setminus A) \geq \lfloor \frac{n}{2} \rfloor + 1$ for every $1 \leq i \leq \lceil \frac{n}{2} \rceil$, and then

$$f(G) \geq w(A, V \setminus A) \geq \lceil \frac{n}{2} \rceil \cdot (\lfloor \frac{n}{2} \rfloor + 1) = \lfloor \frac{n^2}{4} \rfloor + \lceil \frac{n}{2} \rceil$$

Otherwise, for every $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n - k - 2 \cdot t$ we have $w(v_{2t+i}, U) \leq 2 \cdot t + k$ and therefore $w(V \setminus (U \cup A), U) \leq (2{\cdot}t + k){\cdot}|V \setminus (U \cup A)| = (2{\cdot}t + k){\cdot}(\lfloor \frac{n}{2} \rfloor - 2{\cdot}t - k)$. Thus, by this observation and inequality (3) we get:

$$\begin{aligned}
&f(G) \geq w(A, V \setminus A) \\
&= w(A, V \setminus (U \cup A)) + w(V \setminus U, U) - w(V \setminus (U \cup A), U) \\
&\geq \lceil \frac{n}{2} \rceil \cdot (\lfloor \frac{n}{2} \rfloor - 2{\cdot}t - k) + (n - k - 2{\cdot}t){\cdot}(2{\cdot}t + k) + m - ((2{\cdot}t + k){\cdot}(\lfloor \frac{n}{2} \rfloor - 2{\cdot}t - k)) \\
&= m + \lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil \\
&\geq \lfloor \frac{n^2}{4} \rfloor + f(m)
\end{aligned}$$

□

Lemma 3.8 $\sum_{i=1}^{2t} x_i \leq \lfloor \frac{n}{2} \rfloor - k - 2{\cdot}t$.

Proof: For every $1 \leq i \leq 2{\cdot}t$ let u_i be the matched vertex of v_i. Using Lemma 3.5 we get:

$$\begin{aligned}
&k + 2{\cdot}t + \sum_{i=1}^{2t} x_i \leq \\
&\leq k + 2{\cdot}t + \sum_{i=1}^{2t} (\frac{r - a}{2} + w(v_i, u_i)) \\
&= k + 4{\cdot}t + t{\cdot}(r - a) + 2{\cdot}|M|
\end{aligned}$$

As a result of this and the fact that $|M| = \frac{k+a}{2}$ and $t \leq |M|$ we have:

$$\begin{aligned}
&k + 2{\cdot}t + \sum_{i=1}^{2t} x_i \leq \\
&\leq 4{\cdot}k + 3{\cdot}a + \frac{(k + a){\cdot}(r - a)}{2}
\end{aligned}$$

By Lemma 3.7 and by inequality (3) we get that $k + 2 \cdot t + \sum_{i=1}^{2t} x_i \leq (4 + \frac{4}{\sqrt{2}}) \cdot a + \frac{a(1+\frac{1}{\sqrt{2}})(r-a)}{2}$. If $a = \frac{r}{2} + 4$ then the expression $(4 + \frac{4}{\sqrt{2}}) \cdot a + \frac{a(1+\frac{1}{\sqrt{2}})(r-a)}{2}$ attains its maximum value which is $r^2 \cdot \frac{1+\frac{1}{\sqrt{2}}}{8} + r{\cdot}(2 + \sqrt{2}) + 8{\cdot}(1 + \frac{1}{\sqrt{2}})$ which is at most $\lfloor \frac{n}{2} \rfloor$, (since n is sufficiently large). □

As a result of the lemma we have the following corollary:

Corollary 3.9 $\sum_{i=1}^{2t}(d_i - 1) \leq k$.

Proof: Let $A = \{v \in V \setminus U | \ w(v, v_i) = d_i$ for $1 \leq i \leq 2 \cdot t\}$. By Lemma 3.8, $|A| \geq \lceil \frac{n}{2} \rceil$. Let $B \subseteq A$ with $|B| = \lceil \frac{n}{2} \rceil$. Then,

$$f(G) \geq w(B, V \setminus B) = (\sum_{i=1}^{2t}(d_i - 1) + \lfloor \frac{n}{2} \rfloor - k) \cdot \lceil \frac{n}{2} \rceil$$

Thus, if $\sum_{i=1}^{2t}(d_i - 1) > k$ then G is not a counter example to the theorem. □

We are now ready to prove the main theorem. For every $v_i \in U$ put $d_{v_i} = d_i$. For every $v \in V \setminus U$ let $d_v = 1$. Note that by Corollary 3.9 and by Lemma 3.6 we have $\sum_{v \in V} d_v = n$. For every $A \subseteq V$ define $d_A = \sum_{v \in A} d_v$. For every $A, B \subseteq V$ let $C(A, B) = w(A, B) - d_A \cdot d_B$. By Corollary 3.9 and Lemma 3.6 we have $\sum_{u,v \in V} C(u, v) = |E| - \sum_{u,v \in V}(d_v \cdot d_u) \geq m$. Let $B = \{v \in V | \exists u \in V$ s.t $C(u, v) \neq 0\} \cup U$. By Lemma 3.8 we have $|V \setminus B| \geq \lceil \frac{n}{2} \rceil$. Let H be the subgraph of G induced on B. We have $\sum_{u,v \in B} C(u, v) = \sum_{u,v \in V} C(u, v) \geq m$. H with the weight function C is an integer weighted graph and therefore $f(H) \geq f(m)$. Thus H contains a bipartite subgraph with total weight at least $f(m)$, i.e there are two disjoint sets $B_1, B_2 \subseteq B$ with $C(B_1, B_2) \geq f(m)$. Now, $w(B_1, B_2) = C(B_1, B_2) + d_{B_1} \cdot d_{B_2} \geq f(m) + d_{B_1} \cdot d_{B_2}$. Let T be the union of B_1 and $(\lceil \frac{n}{2} \rceil - d_{B_1})$ vertices of $V \setminus B$. Then,

$$\begin{aligned} & f(G) \geq w(T, V \setminus T) = \\ = \ & w(B_1, B_2) + (\lceil \frac{n}{2} \rceil - d_{B_1}) \cdot d_{B_2} + (\lfloor \frac{n}{2} \rfloor - d_{B_2}) \cdot d_{B_1} + (\lceil \frac{n}{2} \rceil - d_{B_1}) \cdot (\lfloor \frac{n}{2} \rfloor - d_{B_2}) \\ = \ & w(B_1, B_2) + \lfloor \frac{n^2}{4} \rfloor - d_{B_1} \cdot d_{B_2} \\ \geq \ & \lfloor \frac{n^2}{4} \rfloor + f(m) \end{aligned}$$

□

4 Concluding Remarks

The proof of the theorem holds only for large n. It seems plausible that its assertion holds also for smaller n. Nevertheless, Theorem 1.3 gives us the ability to find the exact values of $f(p)$ for infinitely many values of p. In all these cases we can even find the exact value of $g(p)$ because $g(p) \geq f(p)$ and we can find a simple graph G with p edges and with $f(G) = f(p)$, concluding that $f(p) \leq g(p) \leq f(G) = f(p)$. The following proposition, illustrating this reasoning, is a consequence of Theorem 1.3 and the fact (proved in [2], [4]) that $g(\binom{n}{2}) = \lfloor \frac{n^2}{4} \rfloor$ for every $n \geq 2$:

Proposition 4.1 *For every sufficiently large n and for all s satisfying $\lfloor \frac{s^2}{4} \rfloor \leq \lceil \frac{n}{2} \rceil$, we have $f(\binom{n}{2} + \binom{s}{2}) = g(\binom{n}{2} + \binom{s}{2}) = \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{s^2}{4} \rfloor$.*

We can also deduce from Theorem 1.3:

Proposition 4.2 *For all sufficiently large n and for all $m < n$ satisfying $\frac{m}{2} + \frac{1+\sqrt{8m+1}}{8} > \lceil \frac{n}{2} \rceil$ we have $f(\binom{n}{2} + m) = g(\binom{n}{2} + m) = \lfloor \frac{(n+1)^2}{4} \rfloor$.*

It would be nice to prove Conjecture 1.1 and Conjecture 1.2 for all n.

Another fact which is a consequence of the proof of Theorem 1.3 is that if we define a function $h(p)$ as we did for g and f, but where the minimum is taken over all multigraphs, then by the observation in the beginning of the proof of Theorem 1.3 we get that $g(p) = h(p)$ for every $p \geq 1$.

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