Piercing d-intervals

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Abstract

A (homogeneous) d-interval is a union of d closed intervals in the line. Using topological methods, Tardos and Kaiser proved that for any finite collection of d-intervals that contains no $k+1$ pairwise disjoint members, there is a set of $O(d^2k)$ points that intersects each member of the collection. Here we give a short, elementary proof of this result.

A *(homogeneous) d-interval* is a union of d closed intervals in the line. Let $\mathcal{H}$ be a finite collection of d-intervals. The *transversal number* $\tau(\mathcal{H})$ of $\mathcal{H}$ is the minimum number of points that intersect every member of $\mathcal{H}$. The *matching number* $\nu(\mathcal{H})$ of $\mathcal{H}$ is the maximum number of pairwise disjoint members of $\mathcal{H}$. Gyárfás and Lehel [3] proved that $\tau \leq O(\nu^{d!})$ and Kaiser [4] proved that $\tau \leq O(d^2\nu)$. His proof is topological, applies the Borsuk-Ulam theorem and extends and simplifies a result of Tardos [5]. Here we give a very short, elementary proof of a similar estimate, using the method of [2].

Theorem 1 *Let $\mathcal{H}$ be a finite family of d-intervals containing no $k+1$ pairwise disjoint members. Then $\tau(\mathcal{H}) \leq 2d^2k$.*

Proof. Let $\mathcal{H}'$ be any family of d-intervals obtained from $\mathcal{H}$ by possibly duplicating some of its members, and let n denote the cardinality of $\mathcal{H}'$. Note that $\mathcal{H}'$ contains no $k+1$ pairwise disjoint members. Therefore, by Turán's Theorem, there are at least $n(n-k)/(2k)$ unordered intersecting pairs of members of $\mathcal{H}'$. Each such intersecting pair supplies at least 2 ordered pairs (p, I), where p is an end point of one of the intervals in a member of $\mathcal{H}'$, I is a different member of $\mathcal{H}'$, and p lies in I. Since there are altogether at most $2dn$ possible choices for p, there is such a point that lies in at least $\frac{n(n-k)}{k2dn}$ members of $\mathcal{H}'$ besides the one in which it is an endpoint of an interval, showing that there is a point that lies in at least $\frac{n}{2dk}$ of the members of $\mathcal{H}'$. This implies that for any rational weights on the members of $\mathcal{H}$ there is a point that lies in at least a fraction $\frac{1}{2dk}$ of the total weight. By the min-max theorem it follows that there is a collection of m points so that each member of $\mathcal{H}$ contains at least $m/(2dk)$ of them, and thus contains an interval that contains at least $m/(2d^2k)$ of the points. Order the points from left to right, and take the set of all points whose rank in this ordering is divisible by $\lceil m/(2d^2k)\rceil$. This is a set of at most $2d^2k$ points that intersects each member of $\mathcal{H}$, completing the proof. $\square$

Remarks. It may be possible to improve the constant factor in the above proof. Kaiser's estimate is indeed better by roughly a factor of 2; $\tau(\mathcal{H}) \leq (d^2-d+1)\nu(\mathcal{H})$. It will be interesting to decide if the quadratic dependence on d is indeed best possible. Higher dimensional extensions are possible, using the techniques in [2], [1].

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References

[1] N. Alon and G. Kalai, *Bounding the piercing number*, Discrete and Computational Geometry 13 (1995), 245-256.

[2] N. Alon and D. J. Kleitman, Piercing convex sets and the Hadwiger Debrunner (p, q)-problem, *Advances in Mathematics* 96 (1992), 103-112.

[3] A. Gyárfás and J. Lehel, A Helly-type problem in trees, in *Combinatorial Theory and its Applications*, Eds. P. Erdős, A. Rényi and V.T. Sós, North-Holland, Amsterdam 1970, 571–584.

[4] T. Kaiser, Transversals of d-intervals, *Discrete and Computational Geometry*, to appear.

[5] G. Tardos, Transversals of 2-intervals, a topological approach, *Combinatorica* **15**, 1 (1995), 123–134.