EIGENVALUES OF $K_{1,k}$-FREE GRAPHS AND THE CONNECTIVITY OF THEIR INDEPENDENCE COMPLEXES

RON AHARONI, NOGA ALON, AND ELI BERGER

ABSTRACT. Let G be a graph on n vertices, with maximal degree d, and not containing $K_{1,k}$ as an induced subgraph. We prove:

(1) $\lambda(G) \leq (2 - \frac{1}{2k-2} + o(1))d$
(2) $\eta(\mathcal{I}(G)) \geq \frac{k-1}{2k-3}\frac{n}{d}$.

Here $\lambda(G)$ is the maximal eigenvalue of the Laplacian of G, $\mathcal{I}(G)$ is the independence complex of G, and $\eta(\mathcal{C})$ denotes the topological connectivity of a complex $\mathcal{C}$.

The above results supply improved bounds for the existence of independent transversals in $K_{1,k}$-free graphs

1. The maximum Laplace eigenvalue of $K_{1,k}$-free graphs

Let $G = (V, E)$ be a connected graph on the set of vertices $V = \{1, 2, \ldots, n\}$ with maximum degree d. The Laplacian of G is the n by n matrix $L = (L_{ij})$ where $L_{ii} = d_i$ is the degree of the vertex i, $L_{ij} = -1$ if $ij \in E$ and $L_{ij} = 0$ if $i \neq j, ij \notin E$. Let $\lambda = \lambda(G)$ denote the largest eigenvalue of L. It is easy to prove and well known that $\lambda(G) \leq 2d$, and equality holds iff G is d-regular and bipartite. If G contains no induced copy of $K_{1,k}$ this estimate can be improved, as stated in the next theorem.

Theorem 1.1. *Let $G = (V, E)$ be a (simple) graph with maximum degree d containing no induced copy of $K_{1,k}$. Let $t(d, k)$ denote the minimum possible number of edges of a graph on d vertices with no independent set of size k. Then $\lambda(G) \leq 2d - \frac{t(d,k)}{d-1}$.*

Note that by Turán's Theorem $t(d, k) = (1 + o(1))\frac{d^2}{2k-2}$, where the $o(1)$-term tends to zero as d tends to infinity, and thus for large d the above theorem provides an upper bound of $[2 - \frac{1}{2k-2} + o(1)]d$ for $\lambda(G)$. Note also that this is not very far from being tight. Indeed, consider a graph H' obtained from a $(k - 1)$ regular bipartite graph H by replacing each vertex u of H by a clique V_u of size s and by replacing each edge uv of H by a complete bipartite graph connecting each vertex of V_u with each vertex of V_v. This graph is $d = ks - 1$ regular and contains no induced $K_{1,k}$. The vector assigning value 1 to each vertex of H' that belongs to V_u for some u in the first color class of H, and value -1 to each vertex of H' that belongs to V_v for some v in the second color class of H is an eigenvector of the Laplacian of H corresponding to the eigenvalue $ks - 1 + (k - 1)s - (s - 1) = (2k - 2)s > [2 - \frac{2}{k}]d$.

Proof of Theorem 1.1: Let $G = (V, E)$ be a graph with maximum degree d and no induced copy of $K_{1,k}$, and let λ be the largest eigenvalue of the Laplacian L of G. Put $V = \{1, 2, \ldots, n\}$ and let $(x_1, x_2, \ldots, x_n)$ be an eigenvector for the eigenvalue λ, where $\sum_{i=1}^{n} x_i^2 = 1$. Therefore $Lx = \lambda x$ and $x^t Lx = \lambda \|x\|_2^2$. It is easy to

The research of the first author was supported by BSF grant no. 2006099, by an ISF grant and by the Discount Bank Chair at the Technion.

The research of the second author was supported by an ERC advanced grant, by a USA-Israeli BSF grant, and by the Israeli I-Core program.

The research of the third author was supported by BSF grant no. 2006099 and by an ISF grant.

check that $x^tLx = \sum_{ij\in E}(x_i - x_j)^2$ and thus it follows that if d_i is the degree of vertex number i then

$$2d-\lambda = (2d-\lambda)\sum_{i=1}^n x_i^2 = \sum_{i=1}^n (2d-2d_i)x_i^2 + \sum_{i=1}^n (2d_i-\lambda)x_i^2 = \sum_{i=1}^n (2d-2d_i)x_i^2 + \sum_{ij\in E}[2x_i^2+2x_j^2-(x_i-x_j)^2].$$

Therefore

$$2d-\lambda = \sum_{i=1}^n (2d-2d_i)x_i^2 + \sum_{ij\in E}(x_i+x_j)^2. \tag{1}$$

Let $\mathbf{T}$ be the set of all triangles in G. For each triangle T on the vertices i,j,q define

$$S(T) = (x_i+x_j)^2 + (x_j+x_q)^2 + (x_q+x_i)^2.$$

Clearly

$$S(T) = x_i^2+x_j^2+x_q^2+(x_i+x_j+x_q)^2 \geq x_i^2+x_j^2+x_q^2.$$

Fix a vertex i of G, and let $N = N(i)$ be the set of its d_i-neighbors. Since G contains no induced copy of $K_{1,k}$ the induced subgraph of G on N contains no independent set of size k and thus spans at least $t(d_i,k)$ edges. It follows that i is contained in at least $t(d_i,k)$ triangles of G. We thus conclude that

$$\sum_{T\in\mathbf{T}} S(T) \geq \sum_{i=1}^n t(d_i,k)x_i^2. \tag{2}$$

On the other hand, since G has maximum degree d, every edge is contained in at most $(d-1)$ triangles, and therefore

$$\sum_{T\in\mathbf{T}} S(T) \leq (d-1)\sum_{ij\in E}(x_i+x_j)^2. \tag{3}$$

By (2) and (3)

$$(d-1)\sum_{ij\in E}(x_i+x_j)^2 \geq \sum_{i=}^n t(d_i,k)x_i^2,$$

and therefore, by (1),

$$2d-\lambda \geq \sum_{i=1}^n [2(d-d_i) + \frac{t(d_i,k)}{d-1}]x_i^2 \geq \sum_{i=1}^n \frac{t(d,k)}{d-1}x_i^2 = \frac{t(d,k)}{d-1},$$

here we used the fact that $2(d-d_i)(d-1) \geq t(d,k) - t(d_i,k)$ for all $d_i \leq d$. One way to verify that this inequality holds (with room to spare) is to observe that the Turán graph $T(d_i,k)$ with d_i vertices and $t(d_i,k)$ edges is an induced subgraph of $T(d,k)$, which has $t(d,k)$ edges. Therefore one can get $T(d,k)$ from $T(d_i,k)$ by adding vertices one by one, where each new vertex is adjacent to a subset of the existing ones, and hence the number of edges added per added vertex never exceeds $d-1 \leq 2(d-1)$. This completes the proof. $\square$

2. The connectivity of the independence complex of a $K_{1,k}$-free graph.

A simplicial complex $\mathcal{C}$ is called (homotopically) *k-connected* if for every $-1 \leq j \leq k$, every continuous function $f : S^j \to ||\mathcal{C}||$ can be extended to a continuous function $\tilde{f} : B^{j+1} \to ||\mathcal{C}||$ (here $||\mathcal{C}||$ is the underlying space of the geometric realization of $\mathcal{C}$). Intuitively, this means that there is no hole of dimension $k+1$ or less. The *connectivity* $\eta(\mathcal{C})$ of $\mathcal{C}$ is the largest k for which $\mathcal{C}$ is k-connected, plus 2 (this differs from the ordinary definition of connectivity, in which the 2 is not added. The addition of 2 simplifies the statements of the theorems). Another version of connectivity is the homological connecivity: $\eta_H(\mathcal{C})$ is the maximal k such that $H_i(\mathcal{C}) = 0$ for all $i \leq k-2$. It is known that $\eta_H(\mathcal{C}) \geq \eta(\mathcal{C})$, and that they are equal if $\eta(\mathcal{C}) \geq 3$.

The complex of independent sets of vertices in a graph G is denoted by $\mathcal{I}(G)$. In [1] the following lower bound on $\eta(\mathcal{I}(G))$ was proved:

Theorem 2.1. $\eta_H(\mathcal{I}(G)) \geq \frac{|V(G)|}{\lambda(G)}$

This yields, among other things, the following:

Corollary 2.2. *For any graph G with maximum degree $\Delta(G)$, $\eta_H(\mathcal{I}(G)) \geq \frac{|V|}{2\Delta(G)}$.*

The corollary was proved in [10] by simpler methods, and for η it was proved in [3].

Combining Theorem 1.1 with Theorem 2.1 yields lower bounds on $\eta_H(\mathcal{I}(G))$ for any $K_{1,k}$-free graph G. In the present section we shall improve these bounds, using a different method.

For a graph G and a vertex $v \in V(G)$ we denote by $G - v$ the graph obtained from G by the removal of v and the edges adjacent to it, and by $G \wr v$ the graph $G - v - N(v)$, where $N(v)$ denotes the set of neighbors of v. Then $\mathcal{I}(G \wr v) = link_{\mathcal{I}(G)}(v)$ (namely, the complex consisting of those sets whose union with v belongs to $\mathcal{I}(G)$). A standard application of the exactness of the Mayer-Vietoris sequence yields:

Lemma 1. *For every complex $\mathcal{C}$ and vertex v it is true that*

$$\eta_H(\mathcal{C}) \geq \min(\eta_H(\mathcal{C} - v), \eta_H(link_{\mathcal{C}}(v)) + 1). \tag{4}$$

Here is a short explanation why this is true. By the exactness of the Mayer-Vietoris sequence, we have for every pair A, B of complexes

$$\eta_H(A \cup B) \geq \min(\eta_H(A), \eta_H(B), \eta_H(A \cap B) + 1) \tag{5}$$

Let $A = \mathcal{I}(G - v)$ and $B = link_{\mathcal{C}}(v) * \{v\}$ (here "*" denotes the join operation, so $B = link_{\mathcal{C}}(v) \cup \{I + v \mid I \in link_{\mathcal{C}}(v)\}$). Then $A \cap B = link_{\mathcal{C}}(v)$. Clearly, $\mathcal{I}(G) = A \cup B$. Since B is contractible to v, we have $\eta_H(B) = \infty$, and hence (4) follows from (5).

By (4), for any graph G and vertex $v \in V(G)$ the following is true: :

$$\eta_H(\mathcal{I}(G)) \geq \min(\eta_H(\mathcal{I}(G - v)), \eta_H(\mathcal{I}(G \wr v)) + 1). \tag{6}$$

Inequality (5) can be proved also for homotopical η, directly from the definitions. Hence we also have:

Lemma 2. $\eta(\mathcal{I}(G)) \geq \min(\eta(\mathcal{I}(G - v)), \eta(\mathcal{I}(G \wr v)) + 1)$.

A lower bound on η obtained from this inequality can be formulated in terms of a game between two players, CON and NON, on the graph G. CON wants to show high connectivity, NON wants to thwart this attempt. At each step, CON chooses a vertex v or an edge e from the graph remaining at this stage, the starting point being the graph G. NON can then either remove the offered vertex or edge from the graph (we call such a step "deletion"), or remove it and its neighbors (we call such a step "explosion"). The payoff of a game to CON is the number of explosions, or ∞ if there appears at some stage an isolated vertex. We define $\Psi(G)$ to be the maximum, over all strategies of CON, of the minimal payoff. The bound on η is then stated as:

Theorem 2.3. $\eta(\mathcal{I}(G)) \geq \Psi(G)$.

Remark 2.4. A similar result was proved by Meshulam. For an edge $e = uv$ denote by $G \wr e$ the graph $(G \wr u) \wr v$.

Theorem 2.5. [9] *For any edge e:*

$$\eta(I(G)) \geq \min(\eta(I(G - e)), \eta(I(G \wr e)) + 1).$$

This means that in the above game one can also offer NON edges, alongside vertices. There is no example known in which it is provably not enough to offer CON edges to obtain the best bound. Thomasse and Rao [11] gave an example in which it is not enough to use vertex offers.

Theorem 2.6. *If G is a $K_{1,k}$-free graph on n vertices with maximum degree d then $\eta(I(G)) \geq \frac{k-1}{2k-3}\frac{n}{d}$.*

Proof. Let u be a vertex of degree d, and let N be its neighborhood. Choose inductively vertices $v_1, \ldots, v_d$ in N, so that v_i has maximal degree in $G_i = G[N - \{v_j \mid j < i\}]$. Since G is $K_{1,k}$-free, $\alpha(G[N]) < k$, and since $\alpha \geq \frac{|V|}{\Delta+1}$ in any graph with maximum degree Δ, this implies that $deg_{G_i}(x_i) \geq \frac{d-i}{k-1}$ (note that the maximum degree of $G[N]$ is at most $d-1$, since vertices in N are connected in G to u). Play now the game by offering NON one by one the vertices $z_i = v_{d-i+1}$. Not wishing to isolate u, NON will explode one of them, say z_p. By the above, the number of vertices removed by this explosion, plus the number of vertices deleted by NON up to the p-th stage, is at most $p + d - \frac{p}{k-1} \leq \frac{2k-3}{k-1}d$. We have thus forced NON to perform one explosion, and paid the price of removal of at most $\frac{2k-3}{k-1}d$ vertices. Repeating this procedure until the graph is exhausted (or an isolated vertex appears) shows that $\Psi(G) \geq \frac{n}{d}\frac{k-1}{2k-3}$.

□

Remark 2.7. This inequality was proved in [6] for $k = 3$, namely claw free graphs. In line graphs, a subclass of the claw free graphs, a better bound applies. If $G = L(H)$ we get from Theorem 2.1 that $\eta(\mathcal{I}(G)) \geq \frac{|V(G)|}{2\Delta(H)}$, which is also implicit in [4].

Conjecture 2.8. *If $G = (V, E)$ is $K_{1,k}$-free and has maximum degree d then $\eta(I(G)) \geq \frac{|V|}{d+k-1}$.*

If true, then this conjecture is sharp when d is divisible by $k-1$, as shown by taking G to be a Turan graph, the complement of the disjoint union of $\frac{d}{k-1} + 1$ cliques of size $k-1$. Here $\eta(\mathcal{I}(G)) = 1$, namely $\mathcal{I}(G)$ is disconnected.

3. Applications to independent transversals

Let $G = (V, E)$ be a graph and let $V = V_1 \cup V_2 \ldots \cup V_m$ be a partition of V into pairwise disjoint sets. An *independent transversal* in G with respect to this partition is an independent set of G containing exactly one vertex in each V_i.

In [4] the following topological version of Hall's theorem was proved:

Theorem 3.1. *If $\eta(\mathcal{I}(G[\bigcup_{i\in I}])) \geq |I|$ for every $I \subseteq [m]$ then there exists an independent transversal.*

The homological version of this theorem (namely, with η_H replacing η) was proved in [9]. Let $f = \min\{|V_i| : i \leq m\}$. Theorem 3.1 and Corollary 2.2 yield together that, denoting $\Delta(G)$ by d, if $f \geq 2d$ then there exists an independent transversal. This was proved with $f = O(d)$ in [5] and improved in several subsequent papers culminating in [7] and [8] where it is shown that the best possible value of f is $2d$. Theorem 2.1 yields that if the largest eigenvalue of the Laplacian of every induced subgraph of G is bounded by h, then sets V_i of size h suffice. Theorem 1.1 therefore implies that for graphs that contain no induced copy of $K_{1,k}$ sets V_i of size $(2 - \frac{1}{2k-2} + o(1))d$ suffice. A better estimate follows from Theorem 2.6, which shows that in fact even sets of size $\frac{2k-3}{k-1}d = (2 - \frac{1}{k-1})d$ suffice.

Theorem 2.1 can be applied to line graphs of r-uniform simple hypergraphs (that is, hypergraphs in which no two edges share more than one common vertex). For such line graphs, the most negative eigenvalue of the adjacency matrix is at least $-r$, as the adjacency matrix can be written as $BB^t - rI$, where B is the incidence matrix of the hypergraph. This, therefore, implies, by the above reasoning, that any partition into sets V_i of size at least $d + r$, where d is the maximum degree of the line graph, admits an independent transversal. In particular this applies to any partition of the triangles of a Steiner Triple System on n vertices (and $n(n-1)/6$ triangles) into sets of size at least $3n/2 + O(1)$. It seems plausible that the constant $3/2$

here can be reduced, possibly even to 1/2. A similar question regarding line graphs of simple graphs is worth studying as well.

We close this short paper with two questions.

(1) (Improving the estimate in Theorem 1.1) Is it true that in a $K_{1,k}$-free graph with maximum degree d the maximum Laplace eigenvalue is no larger than $(2-\frac{2}{k}+o(1))d$? As mentioned after the statement of Theorem 1.1, this estimate, if correct, is tight.

(2) Do the results that follow for the existence of independent transversals for $K_{1,k}$-free graphs hold also for graphs that contain no induced copy of $K_{k,k}$? In [2] it was shown that for such graphs $\eta(\mathcal{I}(G)) \geq \frac{|V(G)|}{2d-1}$, implying that if $V(G)$ is partitioned into sets of size $2d-1$ then there exists an independent transversal.

References

[1] R. Aharoni, E. Berger and R. Meshulam, Eigenvalues and homology of flag complexes and vector representations of graphs, *Geom. Funct. Anal.* **15** (2005), no. 3, 555–566.
[2] R. Aharoni, R. Holzman, D. Howard and P. Sprüssel, Topological connectivity, cooperative colorings and ISRs, *submitted.*
[3] R. Aharoni, M. Chudnovsky and A. Kotlov, Triangulated spheres and colored cliques, *Disc. Comp. Geometry* **28** (2002), 223–229.
[4] R. Aharoni and P. Haxell, Hall's theorem for hypergraphs, *J. of Graph Theory* **35** (2000), 83–88.
[5] N. Alon, The linear arboricity of graphs, *Israel J. Math.*, **62** (1988), 311–325.
[6] A. Engström, Independence complexes of claw-free graphs, *European J. Combin.* **29** (2008), no. 1, 234–241.
[7] P. Haxell, A condition for matchability in hypergraphs, *Graphs and Combinatorics* **11** (1995), 245 – 248.
[8] P. Haxell and T. Szabó, Odd independent transversals are odd, *Combin. Probab. Comput.* **15** *(2006)*, no. 1-2, 193–211.
[9] R. Meshulam, The clique complex and hypergraph matching, *Combinatorica* **21** (2001), 89–94.
[10] R. Meshulam, Domination numbers and homology, *Journal of Combinatorial Theory Ser. A.*, **102** (2003), 321–330.
[11] S. Thomasse and M. Rao, private communication.

Department of Mathematics, Technion

E-mail address, Ron Aharoni: raharoni@gmail.com

Department of Mathematics, Tel Aviv University

E-mail address, Noga Alon: nogaa@tau.ac.il

Department of Mathematics, University of Haifa

E-mail address, Eli Berger: berger@math.haifa.ac.il