Dense uniform hypergraphs have high list chromatic number

Noga Alon* and Alexandr Kostochka†

October 29, 2010

Mathematics Subject Classification: 05C15, 05C35.
Keywords: List coloring, hypergraph, co-degree

Abstract

The first author showed that the list chromatic number of every graph with average degree d is at least $(0.5 - o(1)) \log_2 d$. We prove that for $r \geq 3$, every r-uniform hypergraph in which at least half of the $(r-1)$-vertex subsets are contained in at least d edges has list chromatic number at least $\frac{\ln d}{100r^3}$. When r is fixed, this is sharp up to a constant factor.

1 Introduction

A *list* for a hypergraph G is an assignment L that provides a subset L_v of a set S (called *the set of colors*) to every vertex v of G. A list L for a hypergraph G is an *s-list* if $|L(v)| = s$ for every $v \in V(G)$. Given a list L for G, an *L-coloring* of G is a proper (that is, with no monochromatic edges) coloring f of the vertices of G such that $f(v) \in L_v$ for every $v \in V(G)$. The *list chromatic number* (or *choice number*) $\chi_\ell(G)$ of a hypergraph G is the minimum integer s such that for every s-list L for G, there exists an L-coloring of G. These notions were introduced (for graphs) independently by Vizing in [10] and by Erdős, Rubin and Taylor in [5]. It turned out that list coloring possesses several properties different from those of an ordinary coloring. Indeed, $\chi_\ell(G)$ can be much larger than the ordinary chromatic number, $\chi(G)$. In particular, as shown in [5] (see also [3] for some extensions), for every $m \geq 1$, $\chi(K_{m,m}) = 2$, but

$$(1 - o(1)) \log_2 m \leq \chi_\ell(K_{m,m}) \leq (1 + o(1)) \log_2 m. \tag{1}$$

We also know that Hadwiger's Conjecture fails for list chromatic number and that list-k-critical graphs may have cut vertices.

An additional property of list coloring of graphs, that is not shared by ordinary vertex coloring, is the result proved by the first author in [1, 2] that the list chromatic number of any (simple) graph with a large average degree is large.

*Sackler School of Mathematics and Blavatnik School of Computer Science, Tel Aviv University, Tel Aviv 69978, Israel and Institute for Advanced Study, Princeton, New Jersey, 08540, USA. E-mail address: nogaa@tau.ac.il. Research supported in part by an ERC Advanced grant, by a USA-Israeli BSF grant, by the Oswald Veblen Fund and by the Bell Companies Fellowship.

†Department of Mathematics, University of Illinois, Urbana, IL, 61801, USA and Sobolev Institute of Mathematics, Novosibirsk, Russia. E-mail address: kostochk@math.uiuc.edu. Research supported in part by NSF grant DMS-0965587 and by grant 09-01-00244 of the Russian Foundation for Basic Research.

Theorem 1 (Alon [2]) *The list chromatic number of each graph with average degree d is at least $(\frac{1}{2} - o(1))\log_2 d$, where the $o(1)$-term tends to zero as d tends to infinity.*

Ramamurthi [9] asked whether a similar statement holds for r-uniform hypergraphs (r-graphs, for short). However, for $r \geq 3$, there is no nontrivial lower bound on the list chromatic number of an r-graph in terms of its average degree.

Example: Let $F_{k,d}$ denote the $2kd$-vertex graph which is the disjoint union of k copies of $K_{d,d}$. Let $G(k,d)$ be the r-uniform hypergraph whose edges are the r-tuples containing at least one edge of $F_{k,d}$. Since $\chi_\ell(G(k,d)) \leq \chi_\ell(F_{k,d})$, by (1), $\chi_\ell(G(k,d)) \leq (1+o(1))\log_2 d$. Note that every $(r-1)$-tuple of vertices in $G(k,d)$ not containing an edge of $F(k,d)$ is contained in at least d edges of $G(k,d)$, and every $(r-1)$-tuple containing an edge of $F_{k,d}$ is contained in $n-r+1$ edges of $G(k,d)$. In particular,

$$\frac{|E(G(k,d))|}{|V(G(k,d))|} \geq \frac{d}{r(2kd)}\binom{2kd}{r-1},$$

and for every given d this ratio can be made arbitrarily large by taking a sufficiently large k.

In view of this example, we considered lower bounds on $\chi_\ell(G)$ in terms of codegrees of $(r-1)$-tuples of vertices. The main result of this paper is the following.

Theorem 2 *Let $d, r \geq 3$. Let $G = (V,E)$ be an r-uniform hypergraph with n vertices and such that for at least $\frac{1}{2}\binom{n}{r-1}$ of $(r-1)$-tuples T of vertices of G, the number of edges of G containing T is at least d. Let $s \leq \frac{\ln d}{100r^3}$ be a positive integer. Then G is not list-s-colorable.*

In view of the example, for fixed r, the bound on s in the theorem is sharp up to a constant factor.

Very recently, several interesting results were obtained on list coloring of simple hypergraphs. Recall that a hypergraph is *simple* if no two of its distinct edges share more than one vertex. Haxell and Pei [6] proved that for every Steiner Triple System T_n on n vertices, $\chi_\ell(T_n) \geq c\frac{\log n}{\log\log n}$. Haxell and Verstraete [7] proved that $\chi_\ell(H) \geq c\sqrt{\log d/\log\log d}$ for every 3-uniform d-regular simple hypergraph H, and independently the authors [4] proved more general and slightly stronger bound: for every fixed $r \geq 3$, $\chi_\ell(H) \geq c(\log d)^{1/(r-1)}$ for each r-uniform simple hypergraph H with average degree d. Conceptually, the last result is more advanced than the result of this paper, but it does not give sharp bounds, as our theorem here does.

In the next three sections we prove Theorem 2. The generic main statement that we will carry by induction is Lemma 3 in Section 3. But first in Section 2 we create conditions which then are used by Lemma 3. Finally, in Section 4 we use the structures provided by Lemma 3 to get the final contradiction. The considerations and procedures in all three sections are similar, but somewhat different.

2 Preliminaries and the first step

If $s = 1$, then the theorem is trivial. So, throughout the paper, $s \geq 2$, $r \geq 3$ and we are working with an r-uniform n-vertex hypergraph $G = (V,E)$ satisfying the conditions of the theorem. We also use the following notation:

$$\text{For } i = 1,\ldots,r-1,\ n_i = \lceil n\ d^{(i-r)/r}\rceil, \text{ and } n_r = n - n_1 - \ldots - n_{r-1}; \qquad t = \lceil d^{1/2r}\rceil. \tag{2}$$

By definition and because $d \le n-r+1 < n$, we have

$$n_1 \ge n^{1/r}, \text{ for } i=1,\dots,r-1,\ 0.5d^{1/r}n_i \le n_{i+1} \le 2d^{1/r}n_i, \text{ and } 0.5d^{(r-i)/r}n_i \le n \le d^{(r-i)/r}n_i. \tag{3}$$

Sometimes the inequality $s \le \frac{\ln d}{100r^3}$ will be used in the form

$$d \ge \exp\{100sr^3\} \ \text{ which is equivalent to } \ d^{1/r} \ge \exp\{100sr^2\}. \tag{4}$$

Claim 1 *Let $G=(V,E)$ satisfy the conditions of the theorem. Then there exists a partition $(V_1,V_2,\dots,V_r)$ of V such that $|V_i| = n_i$ for $i=1,\dots,r$, and the cardinality of the set $W_{2,\dots,r}$ of $(r-1)$-tuples $(v_2,\dots,v_r)$ with the property that $v_i \in V_i$ for $i=2,\dots,r$, and there are at least t vertices $v_1 \in V_1$ such that $(v_1,v_2,\dots,v_r)$ is an edge of G is at least $0.45\prod_{i=2}^r n_i$.*

Proof. Choose $V_1,\dots,V_{r-1}$ as random disjoint subsets of V of the given cardinalities. Say that an ordered $(r-1)$-tuple $(v_2,\dots,v_r)$ with $v_i \in V_i$ for $i=2,\dots,r$ is r*-good*, if it is contained in at least d edges of G. Under the conditions of the theorem, the probability that a random ordered $(r-1)$-tuple $(v_2,\dots,v_r)$ with $v_i \in V_i$ for $i=2,\dots,r$ is r-good is at least $1/2$. The probability that an r-good ordered $(r-1)$-tuple $(v_2,\dots,v_r)$ with $v_i \in V_i$ for $i=2,\dots,r$ has fewer than t vertices $v_1 \in V_1$ such that $(v_1,v_2,\dots,v_r)$ is an edge of G is at most $\sum_{i=0}^{t-1} x_i$, where $x_i = \binom{d}{i}\binom{n-r+1-d}{n_1-i}/\binom{n-r+1}{n_1}$. This sum is at most 0.1. Indeed, for $0 \le i \le t-1$, we have

$$\frac{x_{i+1}}{x_i} = \binom{d}{i+1}\binom{n-r+1-d}{n_1-i-1}\binom{d}{i}^{-1}\binom{n-r+1-d}{n_1-i}^{-1} = \frac{(d-i)(n_1-i)}{(i+1)(n-r-d-n_1+i+2)}$$

$$\ge \frac{(d-d^{1/2r})(n_1-d^{1/2r})}{\lceil d^{1/2r}\rceil n} > \frac{0.5dn_1}{2d^{1/2r}n} \ge \frac{0.25d(nd^{-(r-1)/r})}{d^{1/2r}n} = \frac{1}{4}d^{1/2r} \ge \frac{1}{4}e^{50r^2} > 20.$$

So, since $x_t \le 1$, we have $\sum_{i=0}^{t-1} x_i < 2x_{t-1} < 0.1x_t \le 0.1$. Thus the expectation of $|W_{2,\dots,r}|$ is at least

$$\prod_{i=2}^r n_i \cdot \frac{1}{2} \cdot 0.9 = 0.45\prod_{i=2}^r n_i.$$

Therefore, there exists a suitable partition. □

Fix a partition $(V_1,\dots,V_r)$ of V satisfying Claim 1. Let H_1' be the $(r-1)$-partite hypergraph on $V_2 \cup \ldots \cup V_r$ whose edges are the $(r-1)$-tuples in $W_{2,\dots,r}$.

Let $S=\{1,2,\dots,2rs\}$ be a set of colors. To every $v \in V_1$, assign a subset S_v of s colors from S randomly and independently. Say that an edge $(v_2,\dots,v_r) \in E(H_1')$ is $(r-1)$*-good* if every subset S' of S with cardinality s appears as the list of a vertex $v \in V_1$ such that $(v,v_2,\dots,v_r)$ is an edge of G.

Claim 2 *For every $(v_2,\dots,v_r) \in E(H_1')$, the probability $P_1(v_2,\dots,v_r)$ of the event that $(v_2,\dots,v_r)$ is $(r-1)$-good is at least 0.9.*

Proof. The probability that a given subset S' of S with cardinality s does not appear as the list of a vertex $v \in V_1 \cap N_G(w,u)$ is at most

$$\left(1-\binom{2rs}{s}^{-1}\right)^t \le \exp\{-t/(2re)^s\} \le \exp\{-e^{50sr^2 - s\ln(2re)}\} < \exp\{-e^{40sr^2}\}.$$

Therefore,

$$1 - P_1(v_2, \ldots, v_r) \le \binom{2rs}{s} \exp\{-e^{40sr^2}\} \le \exp\{s \ln(2re) - e^{40sr^2}\} < 0.1,$$

as claimed. $\square$

Fix an assignment L_1 of lists to the vertices in V_1 such that the number of $(r-1)$-good edges in H_1' is at least $0.9|W_{2,\ldots,r}| > 0.4\prod_{i=2}^r n_i$. Let H_1 be the hypergraph formed by the $(r-1)$-good edges in H_1'. As we just derived,

$$|E(H_1)| > 0.4\prod_{i=2}^{r} n_i. \tag{5}$$

Let $W_{3,\ldots,r}$ be the set of the $(r-2)$-tuples $(v_3, \ldots, v_r)$, where $v_i \in V_i$ for $i = 3, \ldots, r$, such that each of them is contained in at least $0.2n_2$ edges of H_1. Since

$$|E(H_1)| \le 0.2n_2 \prod_{i=3}^{r} n_i + 0.8n_2|W_{3,\ldots,r}|,$$

by (5),

$$|W_{3,\ldots,r}| \ge \frac{1}{4}\prod_{i=3}^{r} n_i. \tag{6}$$

Let H_2 be the $(r-2)$-partite hypergraph on $V_3 \cup \ldots \cup V_r$ whose edges are the $(r-2)$-tuples in $W_{3,\ldots,r}$.

Fix an L_1-coloring f_1 of vertices in V_1. For every edge $e \in E(H_1)$, let $X_1(f_1, e)$ be the set of colors in S not used on vertices $v \in V_1$ such that $e \cup \{v\} \in E(G)$. Since e is $(r-1)$-good, $|X_1(f_1, e)| \le s-1$. For every $(v_3, \ldots, v_r) \in E(H_2)$, there exists $X_1(f_1; v_3, \ldots, v_r) \subset S$ with $|X_1(f_1; v_3, \ldots, v_r)| = s-1$ such that at least $0.2n_2\binom{2rs}{s-1}^{-1}$ edges $e \in E(H_1)$ containing $\{v_3, \ldots, v_r\}$ have $X_1(f_1, e) \subseteq X_1(f_1; v_3, \ldots, v_r)$. For every $(v_3, \ldots, v_r) \in E(H_2)$, fix such a set $X_1(f_1; v_3, \ldots, v_r)$ and let $Y_2(f_1; v_3, \ldots, v_r)$ denote the set of $v_2 \in V_2$ such that the set $e = \{v_2, v_3, \ldots, v_r\}$ is an edge of H_1 and $X_1(e, f_1) \subseteq X(f_1; v_3, \ldots, v_r)$. By definition,

$$|Y_2(f_1; v_3, \ldots, v_r)| > 0.2n_2\binom{2rs}{s-1}^{-1} \ge 0.2n_2\binom{2rs}{s}^{-1}. \tag{7}$$

For every $v \in V_2$, let $L'(v)$ be a random s-element subset of $S = [2rs]$ chosen uniformly and independently from all other vertices. By (7), for a given $(v_3, \ldots, v_r) \in E(H_2)$, the probability of the event $M(f_1, v_3, \ldots, v_r)$ that some $S' \subset S$ of cardinality s is not a list of any $v_2 \in Y_2(f_1; v_3, \ldots, v_r)$ is at most

$$\binom{2rs}{s}\left(1 - \binom{2rs}{s}^{-1}\right)^{|Y_2(f_1;v_3,\ldots,v_r)|} \le (2re)^s \exp\left\{-0.2n_2\binom{2rs}{s}^{-2}\right\}.$$

Thus the probability of the event $M(f_1)$ that for at least one $(v_3, \ldots, v_r) \in E(H_2)$, the event $M(f_1, v_3, \ldots, v_r)$ occurs is at most $n^{r-2}(2re)^s \exp\left\{-0.2n_2(2re)^{-2s}\right\}$. Since there are at most s^{n_1}

different L_1-colorings of V_1, the probability of the event M_1 that $M(f_1)$ occurs for at least one L_1-coloring f_1, is at most

$$s^{n_1} n^{r-2} (2re)^s \exp\left\{-\frac{n_2}{5}(2re)^{-2s}\right\} = \exp\left\{n_1 \ln s + (r-2)\ln n + s\ln(2re) - \frac{n_2}{5}(2re)^{-2s}\right\}.$$

By (3), the expression in the last exponent is at most

$$n_1 \ln s + (r-2)(\ln n_1^r) + s\ln(2re) - \frac{n_1 e^{100sr^2}}{20} e^{-2s\ln(2re)} \leq$$

$$\leq n_1 \left(\ln s + r^2 \frac{\ln n_1}{n_1} + \frac{s \ln 6r}{n_1} - \exp\{100sr^2 - 2s\ln 6r - \ln 20\}\right) < 0.$$

It follows that with positive probability for every L_1-coloring f_1 of V_1, no event $M(f_1)$ happens. This yields the following claim.

Claim 3 *There exists an assignment L_2' of s-lists from S to vertices in V_2 such that for every L_1-coloring f_1 of V_1 and every $(v_3, \dots, v_r) \in E(H_2)$, there exists $X_1(f_1; v_3, \dots, v_r) \subset S$ with $|X_1(f_1; v_3, \dots, v_r)| = s - 1$ and a set $Y_2(f_1; v_3, \dots, v_r)$ of vertices $v_2 \in V_2$ such that the set $e = \{v_2, v_3, \dots, v_r\}$ is an edge of H_1 and*
(a) for every $v_2 \in Y_2(f_1; v_3, \dots, v_r)$, every color $\alpha \in S - X_1(f_1; v_3, \dots, v_r)$ is used on a vertex $v_1 \in V_1$ such that $(v_1, v_2, \dots, v_r) \in E(G)$;
(b) every $S' \subset S$ of cardinality s is the list of some $v_2 \in Y_2(f_1; v_3, \dots, v_r)$.

We need this to prove the next claim.

Claim 4 *There exists an assignment L_2 of lists to vertices in $V_1 \cup V_2$ such that for every L_2-coloring f_2 of $V_1 \cup V_2$ and every $e = (v_3, \dots, v_r) \in E(H_2)$, there exists $Z_2(f_2; v_3, \dots, v_r) \subset S$ with $|Z_2(f_2; v_3, \dots, v_r)| = 2rs - 2(s-1)$ such that for every $\alpha \in Z_2(f_2; v_3, \dots, v_r)$ there is an edge $(v_1, v_2, v_3, \dots, v_r) \in E(G)$ containing e with $f_2(v_1) = f_2(v_2) = \alpha$.*

Proof. Let

$$L_2(v) := \begin{cases} L_1(v), & \text{if } v \in V_1; \\ L_2'(v), & \text{if } v \in V_2, \end{cases}$$

where L_1 and L_2' are provided by Claims 2 and 3. Let f_2 be an arbitrary L_2-coloring of $V_1 \cup V_2$. Let f_1 denote the restriction of f_2 to V_1. Let $e = (v_3, \dots, v_r) \in E(H_2)$. Let $X_1(f_1; v_3, \dots, v_r) \subset S$ with $|X_1(f_1; v_3, \dots, v_r)| = s - 1$ and $Y_2(f_1; v_3, \dots, v_r)$ be the sets provided by Claim 3. Then by Claim 3(b), the set $F_2(f_1, e) := f_2(Y_2(f_1; v_3, \dots, v_r))$ contains at least $2rs - (s-1)$ colors. So, for each $\alpha \in F_2(f_1, e)$, there exists $v_2(\alpha) \in Y_2(f_1; v_3, \dots, v_r)$ such that $f_2(v_2(\alpha)) = \alpha$. By Claim 3(a), if $\alpha \notin X_1(f_1; v_3, \dots, v_r)$, then there is $v_1 \in V_1$ such that $(v_1, v_2, \dots, v_r) \in E(G)$ and $f_1(v_1) = \alpha$. Thus the claim holds for any subset $Z_2(f_2; v_3, \dots, v_r)$ of the set $F_2(f_1, e) - X_1(f_1; v_3, \dots, v_r)$ with $|Z_2(f_2; v_3, \dots, v_r)| = 2rs - 2(s-1)$. $\square$

3 The general step

Mimicking the proofs of Claims 2-4, we generalize their statements to the following.

Lemma 3 *Let $G = (V, E)$ be a hypergraph as in the statement of the theorem and $(V_1, V_2, \ldots, V_r)$ be a partition of V satisfying Claim 1. Let H_1 be the $(r-1)$-uniform $(r-1)$-partite hypergraph defined in the previous section. For $j = 2, \ldots, r-1$, there exists an $(r-j)$-uniform $(r-j)$-partite hypergraph H_j and a list assignment L_j to the vertices in $V_1 \cup \ldots \cup V_j$ satisfying all the properties below:*
(Q1) for every edge $(v_{j+1}, v_{j+2}, \ldots, v_r) \in E(H_j)$, each v_i belongs to V_i for $i = j+1, \ldots, r$.
(Q2) $|E(H_j)| \geq 2^{-j} \prod_{i=j+1}^{r} n_i$.
(Q3) Each edge of H_j is contained in at least $5^{1-j} n_j$ edges of H_{j-1}.
(Q4) For every L_j-coloring f_j of $\bigcup_{i=1}^{j} V_i$ and every $(v_{j+1}, \ldots, v_r) \in E(H_j)$, there is $Z_j(f_j; v_{j+1}, \ldots, v_r) \subset S$ with $|Z_j(f_j; v_{j+1}, \ldots, v_r)| = 2rs - j(s-1)$ such that for every $\alpha \in Z_j(f_j; v_{j+1}, \ldots, v_r)$ there is an edge $(v_1, v_2, v_3, \ldots, v_r) \in E(G)$ containing e with $f_j(v_1) = f_j(v_2) = \ldots = f_j(v_j) = \alpha$.

Proof. For $j = 2$, the statement follows from (6) and Claims 2, 3 and 4. Suppose that the statement holds for $2 \leq j' \leq j-1$, that H_{j-1} is the corresponding $(r-j+1)$-uniform hypergraph and that L_{j-1} is the corresponding list assignment to the vertices in $V_1 \cup \ldots \cup V_{j-1}$. Let the edges of H_j be the $(r-j)$-tuples $(v_{j+1}, \ldots, v_r)$, where $v_i \in V_i$ for $i = j+1, \ldots, r$, such that each of them is contained in at least $n_j 5^{1-j}$ edges of H_1. So, (Q1) and (Q3) hold by definition. Each edge of H_j is contained in at most n_j edges of H_{j-1} and every other $(r-j)$-tuple $(v_{j+1}, \ldots, v_r)$, with $v_i \in V_i$ for $i = j+1, \ldots, r$ is contained in fewer than $n_j 5^{1-j}$ edges of H_1. Thus, since (Q2) holds for $j-1$, we have

$$n_j |E(H_j)| + n_j 5^{1-j} \prod_{i=j+1}^{r} n_i \geq 2^{1-j} \prod_{i=j}^{r} n_i.$$

Cancelling n_j and simplifying, we obtain

$$|E(H_j)| \geq (2^{1-j} - 5^{1-j}) \prod_{i=j+1}^{r} n_i > 2^{-j} \prod_{i=j+1}^{r} n_i,$$

i.e. (Q2) holds.

Fix an L_{j-1}-coloring f_{j-1} of $\bigcup_{i=1}^{j-1} V_i$. Let $(v_{j+1}, \ldots, v_r) \in E(H_j)$. Since the number of distinct sets $Z_{j-1}(f_{j-1}; v_j, v_{j+1}, \ldots, v_r) \subset S$ with $|Z_{j-1}(f_{j-1}; v_j, v_{j+1}, \ldots, v_r)| = 2rs - (j-1)(s-1)$ such that $(v_j, \ldots, v_r) \in E(H_{j-1})$ is at most $\binom{2rs}{(j-1)(s-1)} < \binom{2rs}{(j-1)s}$, there is a set $Y_j = Y_j(f_{j-1}; v_{j+1}, \ldots, v_r) \subset V_j$ with

$$|Y_j| \geq n_j 5^{1-j} \binom{2rs}{(j-1)s}^{-1} \geq n_j 5^{1-j} (2re)^{-s(j-1)} \tag{8}$$

such that for all $v \in Y_j$, the sets $Z_{j-1}(f_{j-1}; v_j, v_{j+1}, \ldots, v_r)$ are the same.

For every $v \in V_j$, let $L'(v)$ be a random subset of $[2rs]$ chosen uniformly and independently from all other vertices. By (8), the probability that for a given $(v_{j+1}, \ldots, v_r) \in E(H_j)$, some $S' \subset S$ of cardinality s is not a list of any $v_j \in Y_j(f_{j-1}; v_{j+1}, \ldots, v_r)$ is at most

$$\binom{2rs}{s} \left(1 - \binom{2rs}{s}^{-1}\right)^{|Y_j(f_{j-1}; v_{j+1}, \ldots, v_r)|} \leq (2re)^s \exp\left\{-n_j 5^{1-j} (2re)^{-s(j-1)} \binom{2rs}{s}^{-1}\right\}.$$

Thus the probability of the event $M(f_{j-1})$ that for at least one $(v_{j+1},\dots,v_r)\in E(H_j)$, some $S'\subset S$ of cardinality s is not a list of any $v\in Y_j(f_{j-1};v_{j+1},\dots,v_r)$ is at most

$$n^{r-j}(2re)^s\exp\left\{-n_j5^{1-j}(2re)^{-s(j-1)}(2re)^{-s}\right\}.$$

So, since there are $s^{n_1+\ldots+n_{j-1}}\le s^{2n_{j-1}}$ different L_{j-1}-colorings of $\bigcup_{i=1}^{j-1}V_i$, the probability of the event M_{j-1} that $M(f_{j-1})$ occurs for at least one coloring f_{j-1}, is at most

$$s^{2n_{j-1}}n^{r-j}(2re)^s\exp\left\{-n_j5^{1-j}(2re)^{-rs}\right\}=\exp\{2n_{j-1}\ln s+r-j\ln n+s\ln(2re)-n_j5^{1-j}(2re)^{-rs}\}.$$

Similarly to the proof of Claim 3, by (3), the expression in the exponent is at most

$$2n_{j-1}\ln s+(r-2)(\ln n_{j-1}^r)+s\ln(2re)-\frac{n_{j-1}e^{100sr^2}}{2}5^{1-j}e^{-rs\ln(2re)}\le$$

$$\le n_{j-1}\left(2\ln s+r^2\frac{\ln n_{j-1}}{n_{j-1}}+s\frac{\ln(2re)}{n_{j-1}}-e^{100sr^2-j\ln 5-rs\ln(2re)}\right)<0.$$

It follows that with positive probability for every coloring f_{j-1} of $V_1\cup\ldots\cup V_{j-1}$, no event $M(f_1)$ happens. This means that there exists an assignment L'_j of s-lists from $[2rs]$ to the vertices in V_j such that for every L_{j-1}-coloring f_{j-1} of $\bigcup_{i=1}^{j-1}V_i$ and for every $(v_{j+1},\dots,v_r)\in E(H_j)$, there exists a set $X_{j-1}(f_{j-1};v_{j+1},\dots,v_r)\subset S$ with $|Z_{j-1}(f_{j-1};v_{j+1},\dots,v_r)|=(j-1)(s-1)$ and a set $Y_j(f_{j-1};v_{j+1},\dots,v_r)$ of vertices $v_j\in V_j$ such that the set $\{v_j,v_{j+1},\dots,v_r\}$ is an edge of H_{j-1} and

(i) for every $\alpha\in S-X_{j-1}(f_{j-1};v_{j+1},\dots,v_r)$, there is an edge $(v_1,v_2,v_3,\dots,v_r)\in E(G)$ containing $\{v_j,v_{j+1},\dots,v_r\}$ with $f_{j-1}(v_1)=f_{j-1}(v_2)=\ldots=f_{j-1}(v_{j-1})=\alpha$;

(ii) every $S'\subset S$ of cardinality s is the list of some $v_j\in Y_j(f_{j-1};v_{j+1},\dots,v_r)$.

Fix such an assignment L'_j of lists to vertices in V_j and let

$$L_j(v):=\begin{cases}L_{j-1}(v), & \text{if } v\in\bigcup_{i=1}^{j-1}V_i;\\ L'_j(v), & \text{if } v\in V_j.\end{cases}$$

For a coloring f_j of $\bigcup_{i=1}^{j}V_i$, let f_{j-1} denote its restriction to $\bigcup_{i=1}^{j-1}V_i$. Then by (ii), for every $e=(v_{j+1},\dots,v_r)\in E(H_j)$ and every L_j-coloring f_j of $\bigcup_{i=1}^{j}V_i$, set $F_j(f_j,e):=f_j(Y_j(f_{j-1};v_{j+1},\dots,v_r))$ contains at least $2rs-(s-1)$ colors. Let $\alpha\in F_j(f_j,e)-X_{j-1}(f_{j-1};v_{j+1},\dots,v_r)$. Then there is $v_j\in Y_j(f_{j-1};v_{j+1},\dots,v_r)$ with $f_j(v_j)=\alpha$ and by (i), there are $v_1\in V_1,\dots,v_{j-1}\in V_{j-1}$ such that $(v_1,v_2,\dots,v_r)\in E(G)$ and $f_j(v_1)=f_j(v_2)=\ldots=f_j(v_{j-1})=\alpha$. Thus the lemma holds for any subset $Z_j(f_j;v_{j+1},\dots,v_r)$ of the set $F_j(f_j,e)-X_{j-1}(f_{j-1};v_{j+1},\dots,v_r)$ with $|Z_j(f_j;v_{j+1},\dots,v_r)|=2rs-j(s-1)$. $\square$

4 The final step

By Lemma 3 for $j=r-1$, there exists a 1-uniform hypergraph H_{r-1} (in other words, a set W of vertices) and a list assignment L_{r-1} to the vertices in $V_1\cup\ldots\cup V_{r-1}=V-V_r$ satisfying all the properties below:
(Q1) $W\subseteq V_r$.
(Q2) $|W|\ge 2^{-r+1}n_r\ge 2^{-r}n$.

(Q3) Each $v_r \in W$ is contained in at least $5^{2-r} n_{r-2}$ edges of H_{r-2}.
(Q4) For every L_{r-1}-coloring f_{r-1} of $\bigcup_{i=1}^{r-1} V_i$ and every $v_r \in W$, there exists $Z_{r-1}(f_{r-1}, v_r) \subset S$ with $|Z_{r-1}(f_{r-1}, v_r)| = 2rs - (r-1)(s-1)$ such that for every $\alpha \in Z_{r-1}(f_{r-1}, v_r)$ there is an edge $(v_1, v_2, v_3, \ldots, v_r) \in E(G)$ containing e with $f_{r-1}(v_1) = f_{r-1}(v_2) = \ldots = f_{r-1}(v_{r-1}) = \alpha$.

Now we essentially repeat the proof of Lemma 3. Fix an L_{r-1}-coloring f_{r-1} of $V - V_r$. Since the number of distinct sets $Z_{r-1}(f_{r-1}, v_r) \subset S$ with $|Z_{r-1}(f_{r-1}, v_r)| = 2rs - (r-1)(s-1)$ is at most $\binom{2rs}{(r-1)(s-1)} < \binom{2rs}{sr}$, there is a set $Y_r(f_{r-1}) \subset W$ with

$$|Y_r(f_{r-1})| \geq 2^{-r} n \binom{2rs}{sr}^{-1} \geq 2^{-r} n 6^{-rs} \tag{9}$$

such that for all $v_r \in Y_r(f_{r-1})$, the sets $Z_{r-1}(f_{r-1}, v_r)$ are the same.

For every $v \in V_r$, let $L'(v)$ be a random s-list chosen from $[2rs]$ uniformly and independently from all other vertices. By (9), the probability of the event $M(f_{r-1})$ that some $S' \subset S$ of cardinality s is not a list of any $v \in Y_r(f_{r-1})$ is at most

$$\binom{2rs}{s} \left(1 - \binom{2rs}{s}^{-1}\right)^{|Y_r(f_{r-1})|} \leq (2re)^s \exp\left\{-2^{-r} n 6^{-rs} \binom{2rs}{s}^{-1}\right\}.$$

Since there are at most

$$s^{n_1 + \ldots + n_{r-1}} \leq s^{2n_{r-1}}$$

different L_{r-1}-colorings of $V - V_r$, the probability of the event M_{r-1} that $M(f_{r-1})$ occurs for at least one L_{r-1}-coloring f_{r-1}, is at most

$$s^{2n_{r-1}} (2re)^s \exp\left\{-n 2^{-r} (6r)^{-rs} (2re)^{-s}\right\} \leq \exp\{2n_{r-1} \ln s + s \ln(6r) - n 2^{-r} (6r)^{-(r+1)s}\}. \tag{10}$$

Since $n \geq n_{r-1} d^{1/r}/4 \geq n_{r-1} \exp\{100 r^2 s\}/4$, the last expression in (10) is less than 1. So, with positive probability the event M_{r-1} does not hold. It follows that there exists an assignment of lists to the vertices in V_r such that for every L_{r-1}-coloring f_{r-1} of $\bigcup_{i=1}^{r-1} V_i$, there exists a set $X_{r-1}(f_{r-1}) \subset S$ with $|X_{r-1}(f_{r-1})| = (r-1)(s-1)$ and a set $Y_r(f_{r-1})$ of vertices $v_r \in W$ such that

(i) for every $\alpha \in S - X_{r-1}(f_{r-1})$ there is an edge $(v_1, v_2, v_3, \ldots, v_r) \in E(G)$ containing v_r with $f_{r-1}(v_1) = f_{r-1}(v_2) = \ldots = f_{r-1}(v_{r-1}) = \alpha$;

(ii) every $S' \subset S$ of cardinality s is the list of some $v_r \in Y_r(f_{r-1})$. Fix such an assignment L'' of lists to vertices in V_r and let

$$L_r(v) := \begin{cases} L_{r-1}(v), & \text{if } v \in \bigcup_{i=1}^{r-1} V_i; \\ L''(v), & \text{if } v \in V_r. \end{cases}$$

For a coloring f_r of G, let f_{r-1} denote its restriction to $\bigcup_{i=1}^{r-1} V_i$. Then by (ii), for every L_r-coloring f_r of G, the set $F_r(f_r) := f_r(Y_r(f_{j-1}))$ contains at least $2rs - (s-1)$ colors. Let $\alpha \in F_r(f_r) - X_{r-1}(f_{r-1})$. Then by definition there is $v_r \in Y_r(f_{r-1})$ with $f_r(v_r) = \alpha$ and by (i), there are $v_1 \in V_1, \ldots, v_{r-1} \in V_{r-1}$ such that $(v_1, v_2, \ldots, v_r) \in E(G)$ and $f_r(v_1) = f_r(v_2) = \ldots = f_r(v_{r-1}) = \alpha$. So, we have an edge of G all whose vertices are colored with α. This proves the theorem. $\square$

References

[1] N. Alon, Restricted colorings of graphs. In: *Surveys in Combinatorics, Proc. 14th British Combinatorial Conference* Vol. 187 of London Math. Soc. Lecture Notes (K. Walker, ed.), 1993, Cambridge University Press, 1–33.

[2] N. Alon, Degrees and choice numbers, Random Structures & Algorithms **16** (2000), 364–368.

[3] N. Alon and M. Krivelevich, The choice number of random bipartite graphs, *Ann. Comb.* **2** (1998), 291–297.

[4] N. Alon and A. Kostochka, Hypergraph list coloring and Euclidean Ramsey Theory, *Random Structures and Algorithms, to appear.*

[5] P. Erdős, A. L. Rubin and H. Taylor, *Choosability in graphs*, Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium XXVI, 1979, 125-157.

[6] P. E. Haxell and M. Pei, On list coloring Steiner triple systems, J. Combin. Des. 17 (2009), no. 4, 314–322.

[7] P. E. Haxell and J. Verstraete, List coloring hypergraphs, *Elect. J. Combinatorics* 17 (2010), #R129, 12pp.

[8] T. Jensen and B. Toft, *Graph Coloring Problems*, John Wiley and Sons Inc., New York, 1995.

[9] R. Ramamurthi, *Oral communication.*

[10] V. G. Vizing, *Coloring the vertices of a graph in prescribed colors* (in Russian), Diskret. Analiz. No. 29, Metody Diskret. Anal. v. Teorii Kodov i Shem 101 (1976), 3-10.