# Approximating the maximum clique minor and some subgraph homeomorphism problems

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**Abstract.** We consider the "minor" and "homeomorphic" analogues of the maximum clique problem, i.e., the problems of determining the largest $h$ such that the input graph (on $n$ vertices) has a minor isomorphic to $K_h$ or a subgraph homeomorphic to $K_h$, respectively, as well as the problem of finding the corresponding subgraphs. We term them as the *maximum clique minor* problem and the *maximum homeomorphic clique* problem, respectively. We observe that a known result of Kostochka and Thomason supplies an $O(\sqrt{n})$ bound on the approximation factor for the maximum clique minor problem achievable in polynomial time. We also provide an independent proof of nearly the same approximation factor with explicit polynomial-time estimation, by exploiting the minor separator theorem of Plotkin *et al.*
Next, we show that another known result of Bollobás and Thomason and of Komlós and Szemerédi provides an $O(\sqrt{n})$ bound on the approximation factor for the maximum homeomorphic clique achievable in polynomial time. On the other hand, we show an $\Omega(n^{1/2-O(1/(\log n)^{\gamma})})$ lower bound (for some constant $\gamma$, unless $\mathcal{NP} \subseteq \mathrm{ZPTIME}(2^{(\log n)^{O(1)}})$) on the best approximation factor achievable efficiently for the maximum homeomorphic clique problem, nearly matching our upper bound.
Finally, we derive an interesting trade off between approximability and subexponential time for the problem of subgraph homeomorphism where the guest graph has maximum degree not exceeding three and low treewidth. In particular, we show that for any graph $G$ on $n$ vertices and a positive integer $q$ not exceeding $n$, one can produce either an $n/q$ approximation to the longest cycle problem in polynomial time, or find an optimal longest cycle of $G$ in time $2^{O(q\log q+\log n)}$.

## 1 Introduction

Considered as an injective mapping, the *subgraph isomorphism* of $P$ into $G$ consists of a mapping of vertices of $P$ into vertices of $G$ so that edges of $P$ map to corresponding edges of $G$. Generalizations of this mapping include *subgraph homeomorphism*, or equivalently, *topological embedding*, where vertices of $P$ map to vertices of $G$ and edges of $P$ map to vertex-disjoint paths in $G$, and *minor containment*, where vertices of $P$ map to disjoint connected subgraphs of $G$ and edges of $G$ map to edges of $G$.

All these problems are inherently NP-complete when the pattern and guest graphs are not fixed [18]. For fixed $P$, all are solvable in polynomial time, which in case of subgraph homeomorphism and minor containment is highly non-trivial to show [32]. They remain to be NP-complete for several special graph classes, e.g., for graphs of bounded treewidth [20, 29]. Restricting the pattern graph $P$ to complete graphs or simple cycles or paths does not help in the case of subgraph isomorphism. The maximum clique, Hamiltonian cycle and Hamiltonian path problems are well known as basic NP-complete problems [18]. Their optimization versions are also known to be very difficult to approximate. For instance, it is known that unless $\mathcal{NP} \subseteq \mathrm{ZPTIME}(2^{(\log n)^{O(1)}})$, no polynomial-time algorithm for maximum clique (or, equivalently, for maximum independent set) can achieve an approximation factor of $n^{1-O(1/(\log n)^{\gamma})}$ for some constant $\gamma$ [27] (see also [12, 21]) [3]. On the other hand, the best known polynomial-time approximation algorithm for maximum clique achieves solely an $n \log^2 \log n / \log^3 n$ factor [15]. The situation is not better in case of the optimization versions of the Hamiltonian cycle and path problems, called the *longest cycle* and *longest path* problems [14]. For example, the best known polynomial-time approximation algorithm for the longest path problem achieves only $n \log\log n / \log^2 n$ factor [4, 17] [4] The longest path problem cannot be approximated within any constant factor in polynomial time unless $P = NP$ or within any $2^{O(\log^{1-\epsilon} n)}$ factor, where $\epsilon > 0$, in polynomial time, unless $NP \subset DTIME(2^{\log^{1/\epsilon} n})$ [25]. Generally, the directed versions of the longest cycle and longest path problems seem to be even harder (see [4]). Nevertheless, on the positive side, in graphs of maximum degree not exceeding three it is possible to approximate the longest cycle problem within $O(n^{1-(\log_3 2)/2})$ in polynomial time [14]. Furthermore, it is shown in [2] that a path of length $k$ (if it exists) can be found in time $2^{O(k)} n^{O(1)}$ which implies that the longest path problem is fixed-parameter tractable. A similar result is obtained in [2] for all (directed and undirected) graphs of size $k$ and bounded tree width. See also [17] for some related results.

In the first part of this paper, we consider the "minor" analogue of the maximum clique problem, i.e., the problem of determining the largest $h$ such that the input graph has a minor isomorphic to $K_h$. By a known result, obtained independently by Kostochka [26] and Thomason [33] (c.f., also [10]), for every graph with average degree not less than $r\sqrt{\log r}$, $h = \Omega(r)$ holds. We observe that Kostochka's proof of this result provides an $O(\sqrt{n})$ bound on the approximation factor for $h$, i.e., for the maximum clique minor problem, achievable in polynomial time. Interestingly, we also provide an independent proof of nearly the same

[3] An algorithm achieves an approximation factor $f$ or is an $f$-approximation algorithm for a maximization graph problem if for any graph it produces a feasible solution to the problem of size at most $f$ times smaller than the optimum.

[4] In [16], Gabow derives an $n/\exp(\Omega(\sqrt{\log n/\log\log n}))$ approximation factor for the longest path problem by iterating the method of Björklund and Husfeldt from [4]. This has been further improved to $n^{1-\Omega(1/\log\log n)}$ by Feder and Motwani in a forthcoming paper in SODA 2005.

approximation factor with explicit polynomial-time estimation, by exploiting the minor separator theorem of Plotkin *et al.*

In the second part, we consider the maximum homeomorphic clique problem, i.e., the problem of determining the largest $h$ such that the input graph has a subgraph homeomorphic to $K_h$. By another known result, obtained independently by Bollobás and Thomason [9], and by Komlós and Szemerédi [28] (see also [10]), for every graph of average degree not less than $r^2$, $h = \Omega(r)$ holds. We argue that a subgraph homeomorphic to $K_{\Omega(r)}$ can be constructed in polynomial time by following the proof of this result given in [10]. This yields an $O(\sqrt{n})$ approximation factor for the maximum $h$, i.e., maximum homeomorphic clique. On the other hand, we show that the aforementioned results on the approximability of the standard maximum clique problem yield a relatively tight $\Omega(n^{1/2-O(1/(\log n)^\gamma)})$ lower bound for some constant $\gamma$, unless

$$\mathcal{NP} \subseteq \mathrm{ZPTIME}(2^{(\log n)^{O(1)}}),$$

on the approximation factor achievable in polynomial time for the maximum homeomorphic clique problem.

Our results give evidence that the maximum clique minor problem and the subgraph homeomorphism problem might be somewhat easier than the subgraph isomorphism problem. The spectacular result of Robertson and Seymour [32] showing that for any fixed guest graph the minor containment problem is solvable in cubic time implies that it is so called *fixed-parameter tractable* [11]. The maximum clique problem is complete for the so called class $W[1]$ (see [11]). Hence, the maximum clique problem as well as its generalization, the subgraph isomorphism problem, are likely to be fixed-parameter intractable. (On the other hand, the subgraph isomorphism and homeomorphism problems restricted to $k$-connected partial $k$-trees are solvable in polynomial time [29, 19] whereas the minor containment problem is still NP-complete under such restriction [29].)

In the third part, we study the subgraph homeomorphism (sometimes called *topological embedding*) problem for guest graphs of maximum degree not exceeding three and low treewidth, applying, again, the minor separator theorem of Plotkin *et al.* [30] in order to obtain an interesting trade off between approximability and subexponential time. Note that a path or a cycle belongs to this class of graphs. In these two cases, we can obtain the following better results by a more elementary approach (in case of longest path observed by Björklund [5]): For a graph $G$ on $n$ vertices, and a positive integer $q$ smaller than $n$, one can either produce a simple cycle in $G$ of length not less than $q$ in polynomial time, thus yielding an $n/q$ polynomial-time approximation to the longest cycle problem (and a similar approximation to the longest path problem), or find an optimal longest cycle and an optimal longest path of $G$ in time $2^{O(q\log q+\log n)}$. For instance, if we set $q$ to $\lfloor \sqrt{n/\log n} \rfloor$ then we obtain either about $\sqrt{n\log n}$ approximation guarantee in polynomial time or optimal solutions in subexponential time $2^{O(\sqrt{n\log n})}$ for both problems.

Of course, the practical usefulness of this partial result is limited since the potential user cannot choose between these two possibilities. However, this result suggests that perhaps at least one of these possibilities may hold separately. Presently no subexponential algorithms for the longest cycle or path problems are known. In particular, the fastest known algorithm for finding a Hamiltonian cycle in Hamiltonian cubic graphs on $n$ vertices runs in time $O(2^{n/3})$ [13]. Hence, proving the existence of an $n^{1-\epsilon}$ polynomial-time approximation to the longest cycle and path problems, as well as proving the existence of subexponential algorithms for these problems would be surprising and highly interesting results (see [23]).

## 2 Preliminaries

We begin with a formal definition of a (balanced) separator of a graph.

**Definition 1.** *A* $b$-separator *of a weighted graph on $n$ vertices and $m$ edges is a subset $X$ of the vertex set of $G$ whose removal from the graph splits it into connected components, none of which has more than $b$ fraction of the sum the weights of the vertices and edges. The size of the separator is $|X|$. Unless otherwise stated we shall assume the vertices to have weight* 1 *and the edges to have weight* 0.

*Let $k$ be a positive integer. A graph $G$ on $n$ vertices is said to be* $k$-separable *if either it has at most $k+1$ vertices or it has a $\frac{2}{3}$-separator of size at most $k$ whose removal splits $G$ into two $k$-separable subgraphs.*

We shall denote the complete graph on $q$ vertices by $K_q$ and if a graph $G$ has a minor isomorphic to a graph $P$, say that $G$ has a $P$-minor.

The minor separator theorem of Alon *et al.* [1] can be formulated as follows.

**Fact 1** [1]. *There is an algorithm that for a graph $G$ on $n$ vertices and $m$ edges, and an integer $q$, either produces a $K_q$-minor in $G$ or finds a $\frac{2}{3}$-separator of size $O(q^{3/2}\sqrt{n})$ in time $O(\sqrt{qn}(n+m))$.*

Fact 1 has been improved by Plotkin *et al.* for large values of $q$ in [30] as follows.

**Fact 2** [30]. *There is an algorithm that for a graph $G$ on $n$ vertices and $m$ edges, and an integer $q$, either produces a $K_q$-minor in $G$ or finds a $\frac{2}{3}$-separator of size $O(q\sqrt{n\log n})$ in time $O(m\sqrt{n\log n})$.*

The notion of *treewidth* of a graph was originally introduced and investigated by Robertson and Seymour [31] as one of the main contributions in their seminal graph minor project. It has turned out to be equivalent to several other interesting graph theoretic notions like the property of being a partial $k$-tree (see, for example, [3, 6]).

**Definition 2.** *A* tree decomposition *of a graph $G=(V,E)$ is a pair $(\{X_i \mid i \in I\}, T=(I,F))$, where $\{X_i \mid i \in I\}$ is a collection of subsets of $V$, and $T=(I,F)$ is a tree, such that the following conditions hold:*

1. $\bigcup_{i\in I} X_i = V$,
2. *for all edges $(v,w)\in E$, there exists a node $i\in I$, with $v,w\in X_i$, and*

3. *for every vertex $v \in V$, the subgraph of $T$, induced by the nodes $\{i \in I \mid v \in X_i\}$ is connected.*

*Each set $X_i$, $i \in I$, is called the* bag *associated with the $i$th node of the decomposition tree $T$. The* width *of a tree decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ is $\max_{i \in I} |X_i| - 1$. The* treewidth *of a graph is the minimum width of a tree decomposition of it. A* path decomposition *of a graph, the width of a path decomposition and the* pathwidth *of a graph are defined analogously by constraining $T$ to be just a path.*

The following fact follows from the proof of Theorem 20 in [6].

**Fact 3** *Let $G$ be a graph on $n$ vertices. If a sequence of sets on at most $l$ vertices in $G$ satisfying the requirements for the $\frac{2}{3}$-separators in the definition of $l$-separability of $G$ (see Def. 1) is given, then a path decomposition of $G$ of width $O(l \log n)$ can be computed in time $O(nl \log n)$.*

*Proof.* (Sketch.) Let $S$ be the indicated splitting set, and let $G_1$ and $G_2$ be the two subgraphs resulting from removing $S$. Recursively construct path decompositions $(X_1, ..., X_r)$ and $(Y_1, ..., Y_q)$ for $G_1$ and $G_2$, respectively. Then, form the path decomposition $(X_1 \cup S, ..., X_r \cup S, Y_1 \cup S, ..., Y_q \cup S)$ for $G$.

Lemma 16 in [6] yields the next fact.

**Fact 4** *If $H$ is a minor of $G$ then the treewidth of $H$ does not exceed the treewidth of $G$ and the pathwidth of $H$ does not exceed the pathwidth of $G$.*

Theorem 5.2 in [19] yields the following fact.

**Fact 5** *Let $P$ and $G$ be graphs of treewidth $l$, on $n$ vertices totally, given together with their corresponding tree decompositions, and let the maximum degree in $P$ be $O(1)$. One can determine whether or not $P$ can be topologically embedded in $G$, and if so, produce a topological embedding of $P$ in $G$ in time $O(n^{l+2})$.*

In case $P$ is a simple path or a simple cycle, the following parametrized complexity upper bound holds according to [7].

**Fact 6** *For a graph on $n$ vertices given with its tree decomposition of width $l$, the longest path and the longest cycle problem can be solved in time $2^{O(l \log l + \log n)}$.*

## 3 Approximation of maximum $K_q$-minor

By the theorem proved independently by Kostochka [26] and Thomason [33], if a graph has an average degree not less than $r$ then it has a $K_{\Omega(r/\sqrt{\log r})}$-minor. In fact, Kostochka observes that a clique minor of that size can be constructed in polynomial time (see the comment under Theorem 1' in [26]). Hence, we have the following fact.

**Fact 7** *If a graph has an average degree not less than $r$ then a $K_{\Omega(r/\sqrt{\log r})}$-minor in it can be constructed in polynomial time.*

We thus obtain the following theorem.

**Theorem 1.** *There is a polynomial-time $O(\sqrt{n})$-approximation algorithm for the problem of finding the largest $K_q$-minor in a graph on $n$ vertices.*

*Proof.* Let $G$ be a graph on $n$ vertices and let $h$ be the largest $q$ such that $G$ has a $K_q$-minor. We may assume without loss of generality that $h \geq \sqrt{n}$. Note that $G$ must have at least $\binom{h}{2}$ edges, and hence its average degree is $\Omega(h^2/n)$. By Fact 7, we can construct a $K_{\Omega(h^2/(n\sqrt{\log(h^2/n)}\ ))}$-minor which yields the $O(h/(h^2/(n\sqrt{\log(h^2/n)}))$ approximation. By straightforward calculations and $h \geq \sqrt{n}$, this implies the desired $O(\sqrt{n})$ approximation.

In the following, we give an alternative approximation algorithm for the maximum clique minor problem relying on the minor separator theorem of Plotkin *et al.* (Fact 2). It also achieves up to a logarithmic factor the previous $\sqrt{n}$ approximation ratio and its polynomial-time complexity is estimated explicitly.

It is well known that given a tree decomposition $T$ of a graph $G$, for any clique in $G$ there is a bag of $T$ including all of it (see Lemma 4 in [6] and [8]). Hence, the treewidth of $K_q$ is not smaller than $q-1$. Combing this with Fact 4, we obtain immediately the following useful lemma.

**Lemma 1.** *If a graph $G$ has a tree decomposition or a path decomposition of width $l$ then the largest integer $h$ such that $G$ has a $K_h$-minor does not exceed $l+1$.*

By Lemma 1, we obtain the following key lemma.

**Lemma 2.** *There is an algorithm that for a graph $G$ with $n$ vertices and $m \geq n-1$ edges either produces a path decomposition of width $O(q\sqrt{n}\log^{1.5} n)$ or a minor isomorphic to $K_q$ in time $O(mn^{1/2}\sqrt{\log n})$.*

*Proof.* Repeatedly run the algorithm of Plotkin *et al.* from Fact 2 for $K_q$ and $G$ or its subgraphs to produce either a $K_q$-minor or a $\frac{2}{3}$-separator of size $O(q\sqrt{n \log n})$. This gives either a $K_q$-minor, or a path decomposition of $G$ of width $O(q\sqrt{n}\log^{1.5} n)$, by Fact 3. (Being a bit more careful we can in fact shave a logarithjmic factor, as the separators for the small subgraphs are smaller, but we ignore this fact here). We may assume without loss of generality that the algorithm never fails to produce the aforementioned separator since otherwise we obtain a minor of $G$ isomorphic to $K_q$. More exactly, given such a separator, we remove it from the current subgraph of $G$ in order to compute the resulting connected components and group them in two subgraphs, none containing more than two thirds of the vertices of the current subgraph, and then run the algorithm of Plotkin *et al.* on these two subgraphs and so on. By Fact 3, such a sequence of separators yields a path decomposition of width $O(q\sqrt{n}\log^{1.5} n)$. To obtain the time bound it is sufficient to observe that in level number $i$ of the recursion the algorithm of Plotkin *et al.* is run on a set of graphs whose total number of edges is at most $m$, while each of them has at most $(2/3)^i n$ vertices.

We next describe our alternate result on maximum clique minor containment.

**Theorem 2.** *There is an $O(\sqrt{n}\log^{1.5} n)$ approximation algorithm for the problem of finding the largest $K_q$-minor in a graph on $n$ vertices and $m$ edges running in time $O(mn^{1/2}\log^{1.5} n)$.*

*Proof.* Obviously $G$ contains a $K_1 = K_{2^0}$ minor. Run the algorithm in Lemma 2 with $q = 2^i$, $i \geq 1$ until the smallest $i$ such that the algorithm finds a $K_{2^i}$ minor and does not find a $K_{2^{i+1}}$ minor is determined. Then for $q = 2^i$ the algorithm finds a $K_q$-minor, as well as a path decomposition of width $O(2q\sqrt{n}\log^{1.5} n)$. But then the maximum minor in $G$ is of size at most this width plus 1, by Lemma 1.

**Remark**

The last proof can be easily extended to include not necessarily complete guest graphs which are hard to split.

## 4 Approximability of maximum homeomorphic clique

By the theorem proved independently in [9] and [28] (see also [10]), if a graph has an average degree not less than $r^2$ then it has a topological $K_{\Omega(r)}$. In fact, by following the proof of this theorem given in [10], we can observe that such a $K_{\Omega(r)}$-minor can be constructed in polynomial time [5]. Thus, we have the following fact.

**Fact 8** *If a graph has an average degree not less than $r^2$ then a topological $K_{\Omega(r)}$ in it can be constructed in polynomial time.*

This implies the following theorem.

**Theorem 3.** *There is a polynomial-time $O(\sqrt{n})$-approximation algorithm for the problem of finding the largest topological $K_q$-minor in a graph on $n$ vertices.*

*Proof.* Let $G$ be a graph on $n$ vertices and let $h$ be the largest $q$ such that $G$ has a topological $K_q$-minor. As in the proof of Theorem 1, we observe that the average degree of $G$ is $\Omega(h^2/n)$. Hence, by Fact 8, we can construct a topological $K_{\Omega(h/\sqrt{n})}$-minor of $G$ in polynomial time which yields the required $O(h/(h/\sqrt{n})) = O(\sqrt{n})$ approximation.

The following lemma will be useful in proving our lower bound on the approximability of maximum homeomorphic clique.

**Lemma 3.** *There is an algorithm which for a homeomorphic clique of size $h$ in a graph on $n$ vertices determines a clique of size $\Omega(h^2/n)$, contained in the homeomorphic clique, in time polynomial in $n$.*

[5] In particular, a minimum linkage $P$ as in page 172 in the proof in [10] can be found by applying minimum cost flow techniques.

*Proof.* Let $h$ be the number of clique vertices, i.e., endpoints of paths modeling clique edges, in a homeomorphic clique $\tilde{H}$. Note that $\tilde{H}$ can include at most $n-h$ paths having more than one edge directly connecting its clique vertices. Form an auxiliary graph $A$ on the clique vertices of $\tilde{H}$ such that two vertices $u$ and $v$ are connected by an edge if and only if the shortest path in $\tilde{H}$ connecting them has length at least two. Note that $A$ has at most $n-h$ edges and consequently average degree $(n-h)/2h$. Hence, by Turán's Theorem (whose proof is easily seen to be algorithmic, see, for example [22]), one can find an independent set of size $\Omega(h^2/n)$ in $A$ and consequently a clique of size $\Omega(h^2/n)$ in $\tilde{H}$, in polynomial time.

Our lower bound on polynomial-time approximability of maximum homeomorphic clique thus follows from that for maximum clique [27] (see also [12, 21]) by Lemma 3.

**Theorem 4.** *Unless $\mathcal{NP} \subseteq ZPTIME(2^{(\log n)^{O(1)}})$, maximum homeomorphic clique cannot be approximated in polynomial time within a factor $n^{1/2-O(1/(\log n)^{\gamma})}$, for some constant $\gamma$.*

*Proof.* By [27], no polynomial-time algorithm for maximum clique can achieve an approximation factor of $n^{1-O(1/(\log n)^{\gamma})}$ for some constant $\gamma$ unless $\mathcal{NP} \subseteq$ ZPTIME$(2^{(\log n)^{O(1)}})$. Let $x \in O(1/(\log n)^{\gamma})$. It follows that there is no correct polynomial-time approximation algorithm for maximum clique that in case the input graph has a clique of size $> n^{1-x}$ would return a clique of size $\Omega(n^x)$. Suppose that there is a polynomial-time $O(n^{1/2-3x/2})$-approximation algorithm for maximum homeomorphic clique. Let $G$ be the input graph on $n$ vertices. Suppose that $G$ contains a clique of size at least $n^{1-x}$. Then, the aforementioned algorithm would find a homeomorphic clique $\bar{H}$ in $G$ having $\Omega(n^{1/2+x/2})$ clique vertices. It follows by Lemma 3 that one could determine a clique of size $\Omega(n^x)$ in $\bar{H}$, in polynomial time. We obtain a contradiction.

## 5 Subgraph homeomorphism for special guest graphs

We begin by noting that we can use a minor embedding of $K_q$ in a graph to construct a topological embedding of any subgraph of $K_q$ having vertex degrees not exceeding three in the graph.

**Theorem 5.** *Given a graph $G$, its minor isomorphic to $K_q$ and a subgraph $H$ of $K_q$ whose maximum degree is at most three, one can find a topological embedding of $H$ in $G$ in time linear in the size of $G$.*

*Proof.* Let $\phi$ be the mapping from the vertices of $K_q$ to the subsets of the vertex set of $G$ and from the edges of $K_q$ to edges of $G$ that defines the $K_q$-minor of $G$. For each vertex $v$ of $H$, find a spanning tree $T_v$ of the subgraph induced by $\phi(v)$. For each edge $(v, w)$ of $H$, where $(v', w') = \phi(v, w)$, mark $v'$ in $T_v$ and $w'$ in $T_w$. Next, for each vertex $v$ of $H$ prune $T_v$ to the union $U_v$ of the paths in $T_v$

interconnecting at most three marked vertices. It is clear that $U_v$ has the form of either three simple paths meeting at a joint endpoint or just a simple path. By taking the union of the pruned trees $U_v$ over the vertices of $H$ and the $\phi$-images of the edges of $H$, we obtain a subgraph of $G$ homeomorphic with $H$.

By combining Lemma 2 with Theorem 5, we obtain the next theorem.

**Theorem 6.** *Let $1 \le q \le n$ and let $H$ be a subgraph of $K_q$ of maximum degree not exceeding three. There is a polynomial-time algorithm which for any graph $G$ on $n$ vertices produces either a topological embedding of $H$ in $G$ or a path decomposition of $G$ having width $O(q\sqrt{n}\log^{1.5} n)$.*

Fact 5 immediately yields the following lemma.

**Lemma 4.** *Given a graph $G$ on $n$ vertices whose treewidth does not exceed $l$, together with its tree decomposition, and a family $F$ of $k$ graphs of maximum degree $O(1)$ and treewidth not exceeding $l$, each having at most $n$ vertices, one can find a maximum vertex cardinality member of $F$ that can be topologically embedded in $G$ as well as its topological embedding in $G$ in time $O(n^{l+2}k)$.*

By combining Theorem 6 with Lemma 4, we obtain our next main result.

**Theorem 7.** *Let $G$ be a graph on $n$ vertices, let $1 \le q \le n$, and let $F$ be a sequence of graphs $H_i$, $i = 1, ..., n$, where $H_i$ has $i$ vertices, maximum degree at most three and treewidth $O(q\sqrt{n}\log^{1.5} n)$. One can produce either a topological embedding of $H_q$ in $G$ in polynomial time or a maximum vertex cardinality member of $F$ that can be topologically embedded in $G$ together with its topological embedding in $G$ in time $2^{O(q\sqrt{n}\log^{2.5} n)}$.*

Note that in particular simple cycles and simple paths, having treewidth 2 and 1, respectively, satisfy the requirements on the members in the sequence $F$. In these cases, however, one can do better by a similar, yet more elementary approach. Björklund [5] observes that the following is implicit in [7].

Construct a DFS tree for the input graph $G$. Either the length of the deepest path in the tree is at least $q$ or the path decomposition formed by its root-leaf paths has width $q$ and consequently algorithms for longest path in graphs of pathwidth $q$ can be applied to $G$.

Hence, Björklund obtains the following result by Fact 5.

**Fact 9** *Let $G$ be a graph on $n$ vertices, and let $1 \le q \le n$. One can produce either a simple path in $G$ of length at least $q$ in polynomial time, thus yielding an $n/q$ polynomial-time approximation to the longest path problem, or an optimal longest path of $G$ in time $n^{O(q)}$.*

By considering also backward edges in the DFS tree, tree decomposition instead of path decomposition and Fact 6 instead of Fact 5, we obtain the following generalization of Fact 9.

**Theorem 8.** *Let $G$ be a graph on $n$ vertices, and let $1 \leq q \leq n$. One can produce either a simple cycle in $G$ of length at least $q$ in polynomial time, thus yielding an $n/q$ polynomial-time approximation to the longest cycle problem and a similar polynomial-time approximation to the longest path problem, or an optimal longest cycle and an optimal longest path of $G$ in time $2^{O(q \log q + logn)}$.*

To prove the theorem, we need the following simple lemma.

**Lemma 5.** *There is a linear time algorithm that for a graph $G$ and an integer $q \geq 4$, either produces a cycle of length at least $q$ in $G$ or finds its tree decomposition of width not exceeding $q - 2$.*

*Proof.* Clearly we may and will assume that $G = (V, E)$ is connected. Choose arbitrarily a vertex $v$ in $G$, and find, in linear time, a DFS tree $T$ rooted at $v$. Since $G$ is undirected, all the edges of $G$ are either tree-edges or backward edges. If there is a backward edge connecting a vertex $w$ to its ancestor $x$ in $T$ such that the unique path in $T$ from $x$ to $w$ is of length at least $q - 1$, then this edge together with the path give a cycle of length at least $q$ in $G$. Otherwise, form a tree decomposition as follows. The tree of the decomposition is $T$ itself, and the bag $X_u$ for each $u \in V$ is the set of the last $q-1$ vertices on the unique path from the root $v$ to $u$, including $u$ itself (if this path is shorter, the bag contains all its vertices). It is easy to see that in this way we obtain a valid tree decomposition of width $q - 2$.

Lemma 5 combined with Fact 6 yield Theorem 8.

## 6 Final Remark

It is an interesting open problem whether or not a non-trivial lower bound on the approximability of the maximum clique minor problem, possibly even nearly matching our upper bound, could be derived.

In [24], Lund and Yannakakis show that for any non-trivial graph property $\Pi$ which is hereditary on induced subgraphs the problem of finding the maximum number of nodes inducing a subgraph in a directed or undirected graph that satisfies $\Pi$ cannot be approximated within $n^{\epsilon}$, unless $P = NP$. While beeing a clique is such a hereditary property, unfortunately neither beeing a clique minor nor beeing homeomorphic to a clique satisfies this requirement.

### Acknowledgments

We are grateful to Andreas Björklund for bringing Fact 9 to our attention, and to Anders Dessmark and Andrzej Proskurowski for sharing their knowledge on subgraph embedding algorithms for bounded treewidth graphs.

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