Hypergraph list coloring and Euclidean Ramsey Theory

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Abstract

A hypergraph is simple if it has no two edges sharing more than a single vertex. It is s-list colorable (or s-choosable) if for any assignment of a list of s colors to each of its vertices, there is a vertex coloring assigning to each vertex a color from its list, so that no edge is monochromatic. We prove that for every positive integer r, there is a function $d_r(s)$ such that no r-uniform simple hypergraph with average degree at least $d_r(s)$ is s-list-colorable. This extends a similar result for graphs, due to the first author, but does not give as good estimates of $d_r(s)$ as are known for $d_2(s)$, since our proof only shows that for each fixed $r \geq 2$, $d_r(s) \leq 2^{c_r s^{r-1}}$. We use the result to prove that for any finite set of points X in the plane, and for any finite integer s, one can assign a list of s distinct colors to each point of the plane, so that any coloring of the plane that colors each point by a color from its list contains a monochromatic isometric copy of X.

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1 Introduction

1.1 Background

The *list chromatic number* (or *choice number*) $\chi_\ell(G)$ of a graph $G = (V, E)$ is the minimum integer s such that for every assignment of a list L_v of s colors to each vertex v of G, there is a proper vertex coloring of G in which the color of each vertex is in its list. This notion was introduced independently by Vizing in [21] and by Erdős, Rubin and Taylor in [10]. In both papers the authors realized that this is a variant of usual coloring that exhibits several new properties, and that in general $\chi_\ell(G)$, which is always at least as large as the chromatic number of G, may be arbitrarily large even for graphs G of chromatic number 2.

It is natural to extend the notion of list coloring to hypergraphs, and indeed this has been done, among other places, in [20]. The list chromatic number $\chi_\ell(H)$ of a hypergraph H is the minimum

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integer s such that for every assignment of a list of s colors to each vertex of H, there is a vertex coloring of H assigning to each vertex a color from its list, with no monochromatic edges.

An intriguing property of list coloring of graphs, which is not shared by ordinary vertex coloring, is the result proved by the first author in [2, 3] that the list chromatic number of any (simple) graph with a large average degree is large. Indeed, it is shown in [3] that the list chromatic number of any graph with average degree d is at least $(\frac{1}{2} - o(1)) \log_2 d$, where the $o(1)$-term tends to zero as d tends to infinity. Ramamurthi [19] asked whether a similar statement holds for r-uniform hypergraphs (r-graphs, for short). For $r \geq 3$, there is no nontrivial lower bound on the list chromatic number of an r-graph in terms of its average degree. To see this, consider, for example, a perfect matching M consisting of $n/2$ isolated (graph) edges on a set of n vertices, and let $H = H(M)$ be the r-graph on this set of vertices consisting of all r-edges containing at least one edge of M. Then the degree of every vertex of H is $\binom{n-2}{r-2}$, and yet its list chromatic number is 2. More generally, one can replace the matching M by any graph G of list chromatic number bounded by t, and consider the hypergraph H of all r-edges containing at least one edge of G, which clearly satisfies $\chi_\ell(H) \leq t$. For example, if n is divisible by $2g$ and we replace M with the graph G that is the disjoint union of $n/2g$ complete bipartite graphs $K_{g,g}$, then the degree of every vertex of the corresponding r-graph $H = H(G)$ is at least $(1 + o(1))g\binom{n}{r-2}$, where for any fixed r and g, the $o(1)$-term tends to zero as n tends to infinity, whereas the list chromatic number of H is only $(1 + o(1)) \log_2 g$.

1.2 Simple hypergraphs

The dense hypergraphs of bounded list chromatic number in the above examples are not simple. Recall that a hypergraph is called *simple* if every two of its distinct edges share at most one vertex. In the present paper we prove that the result of [3] can be extended to simple r-graphs. This is stated in the following theorem.

Theorem 1.1 *For every fixed $r \geq 2$ and $s \geq 6r$, there is $d = d(r, s)$, such that the list chromatic number of any simple r-graph with n vertices and nd edges is greater than s.*

This extends the main result of [3] (which is the case $r = 2$ of the above theorem), as well as that of Haxell and Pei, who proved in [12] that the list chromatic number of any Steiner Triple System on n vertices is at least s, for all $n \geq n_0(s)$. After obtaining our results we learned that Haxell and Verstraete [13] proved a similar result for the special case of d-regular 3-uniform simple hypergraphs.

It is worth noting that the theorem provides a linear time algorithm for computing, for a given input simple r-graph, a number s such that its list chromatic number is at least s and at most $f(s)$ for some explicit function f. There is no such known result for ordinary coloring, and it is known that there cannot be one under some plausible hardness assumptions in Complexity Theory—see [6] and a few additional related comments in the last section of the present paper.

In order to prove Theorem 1.1, we will prove a stronger statement — if the average degree of an r-graph is sufficiently large, then there is a way to assign lists of s colors, from a set of not too many colors, to all the vertices so that in any vertex coloring from the lists there are many monochromatic edges. This statement seems to be crucial for the proof, and is stated in the following theorem.

Theorem 1.2 *For every fixed $r \geq 2$, there are functions $d_r(s)$, $R_r(s) \geq s$ and $\delta_r(s) \leq 1/R_r(s)$ such that the following holds. For each $s \geq 6r$, every $d > d_r(s)$ and every n-vertex simple r-graph G with dn edges, there is an assignment L of lists of size s from the set $[R_r(s)] = \{1, 2, \ldots, R_r(s)\}$ to the vertices of G such that for every coloring of the vertices of G from these lists, the number of monochromatic edges in G is at least $\delta_r(s) \cdot dn$.*

1.3 A geometric application

A well known problem of Hadwiger and Nelson is that of determining the minimum number of colors required to color the points of the Euclidean plane so that no two points at distance 1 have the same color. Hadwiger showed already in 1945 that 7 colors suffice, and Moser and Moser noted in 1961 that 3 colors do not suffice. These bounds have not been improved, despite a considerable amount of effort by various researchers, see [15, pp. 150-152] and the references therein for more on the history of the problem.

A more general problem is considered in [7], [8], [9], where the main question is the investigation of finite point sets K in the Euclidean space for which any coloring of an Euclidean space of a sufficiently high dimension $d \geq d_0(K, r)$ by r colors must contain a monochromatic copy of K. The main conjecture is that this holds for any set K that can be embedded in a sphere.

The situation is different for list coloring. As described in [16], for every integer s there is an assignment of a list of s colors to each point of the plane such that in any coloring of the plane that colors each point by a color from its list there are two points of distance 1 having the same color. This can be deduced from the main result of [3]. As a corollary of our results here, we prove the following.

Theorem 1.3 *For any finite set X in the Euclidean plane and for any positive integer s, there is an assignment of a list of size s to every point of the plane, such that whenever we color the points of the plane from their lists, there is a monochromatic isometric copy of X.*

1.4 Organization

In the next section we outline the proof of the main result. One of the difficulties in this proof is the problem of handling r-graphs in which the minimum degree is much smaller than the average degree. To overcome this obstacle, we prove in Section 3 a simple decomposition result showing that any r-graph H contains a subgraph H' with at least, say, half the edges of H, that can be decomposed into large r-graphs each having average degree at most r times the minimum degree. The proof of Theorem 1.2 for $r = 2$ is described in Section 4 and the proof for general r in Section 5. Section 6 is devoted to the geometric application, and the final section contains some concluding remarks and open problems.

2 An outline of the proof

The proof of Theorem 1.2 applies induction on r. Although it is possible to start the induction with the trivial case $r = 1$, we prefer to start with $r = 2$, as this supplies a better estimate for the general case. In this section we sketch the proof for the base case $r = 2$ and for the case $r = 3$. The proof of the induction step for general r is similar to the proof for $r = 3$, with a few additional technical complications described in Section 5.

Consider, first, the case $r = 2$. Let $G = (V, E)$ be a graph with n vertices and nd edges. We (try to) start as in [3]: choose a random set B of about $n/\sqrt{d}$ vertices and assign a random list of size s out of a set S of $2s - 1$ colors to each vertex of B. A simple computation shows that with positive (and in fact high) probability many of the vertices v not in B have every subset of size s of S assigned to at least one of their B-neighbors. Fix such a choice of the set B and lists of colors to its vertices. Note now that for each fixed choice of a coloring f of the vertices of B from their lists, at least s distinct colors appear on the B-neighbors of any vertex v of the type mentioned above. If we now assign a random list to such a vertex v, then with probability at least $\binom{2s-1}{s}^{-1} > 4^{-s}$ it will be a forbidden list, that is, it will consist only of colors assigned by f to its neighbors, showing that the coloring f of the B vertices cannot be extended to a proper list coloring of the whole graph. There are only $s^{|B|}$ possible colorings of the vertices of B from their lists, and the probability that no vertex v gets a forbidden list is small enough to ensure that this will not happen for any of these colorings. This argument suffices to show that the list chromatic number of G exceeds s. However, our objective is to prove a stronger result (needed for proceeding with the case of 3-uniform hypergraphs): there is an assignment of lists of size s to the vertices of G such that in any coloring using the lists at least a $\delta_2(s)$-fraction of the edges are monochromatic. With some care, the proof described above does imply that in any such coloring there are at least some $\delta_2(s)$-fraction of the vertices such that for each such vertex v, at least a $\delta_2(s)$-fraction of the edges incident with v are monochromatic. However, since the minimum degree in G may be much smaller than its average degree, this does not suffice.

In order to deal with this case, we first show that G contains a subgraph consisting of at least half the edges of G that can be decomposed into a collection of pairwise edge-disjoint graphs G_i, where each of them is large and its minimum degree (which is also large) is at least half its average degree. It turns out that we can now select one set of vertices B and an assignment of lists of colors to its members so that B and its lists will be good enough to handle simultaneously all graphs G_i. This works because the probabilistic estimates are strong enough to ensure that the events corresponding to all graphs G_i hold. Thus each G_i will contribute its share of monochromatic edges, implying the desired result for $r = 2$.

Given the result for $r = 2$, we sketch the proof of the case $r = 3$, that is, the case of simple 3-graphs. The decomposition result, that holds for 3-graphs as well, enables us to focus on one member of the decomposition, which is a simple 3-graph in which the minimum degree is not much smaller than the average one. Given such a simple 3-graph G with n vertices and nd edges satisfying this condition, we select in it a large number M of random pairwise disjoint sets $V_1, V_2, \ldots, V_M$ of vertices, each of size roughly $\frac{n}{d^{1/4}}$. With high probability many of the vertices v of G not selected to any of these sets, have a (graph)-edge $e_{j,v}$ in each V_j, such that $e_{j,v} \cup v$ is an edge of G. The assumption that G is simple implies that all these 2-edges $e_{j,v}$ are pairwise distinct.

We now apply the induction hypothesis and assign lists of s colors to the vertices in $\cup_j V_j$ so that for each fixed j, in each coloring of these vertices from their lists, a $\delta_2(s)$-fraction of all edges $e_{j,v}$ is monochromatic. The sets of colors used for the different sets V_j are pairwise disjoint, to ensure that a vertex v for which $e_{j,v}$ is monochromatic for many different values of j will have many distinct forbidden colors. A double counting ensures now that in any coloring of the vertices in $\cup_j V_j$ from their lists there is at least a $\delta_3'(s)$-fraction of the vertices not in $\cup_j V_j$ for which there are at least s forbidden colors. We can now assign random lists to all remaining vertices and proceed as in the case $r = 2$ (here, too, the probabilistic estimates are strong enough to handle all hypergraphs in the decomposition simultaneously).

This completes the outline. The details are described in the following three sections.

3 A decomposition result for r-graphs

In this section we prove the following.

Lemma 3.1 *Let $d \geq 4D$. Let $G = (V, E)$ be an r-graph with n vertices and nd edges. Then there is a family of pairwise edge disjoint subgraphs $G_i = (U_i, E_i)$ of G, $1 \leq i \leq p$, satisfying the following five conditions.*
(i) $U_p \subset U_{p-1} \subset U_{p-2} \subset \ldots \subset U_1$,
(ii) $|U_p| \geq \sqrt[r]{nd/4}$,
(iii) The minimum degree of each of the r-graphs G_i is at least D,
(iv) The number of edges of G_i is at most $|U_i| \cdot D$, and
(v) At least half of the edges of G belong to some of the r-graphs G_i.

Proof: Starting with G, as long as there is a vertex of degree less than D in G, omit it until we reach an r-graph with minimum degree at least D. Let U_1 be its vertex set. To construct G_1, for each vertex $u \in U_1$ pick an arbitrary set of D edges of G incident with u, and let all these edges belong to G_1. By construction, G_1 has minimum degree at least D, and has at most $|U_1|D$ edges. We now remove all the edges of G_1 from G and proceed in the same manner. Namely, as long as there is a vertex of degree less than D in (the remaining part of) G, omit it until we reach an r-graph with minimum degree at least D. Let U_2 be its vertex set, and note that $U_2 \subseteq U_1$. For each vertex $u \in U_2$ pick an arbitrary set of D edges incident with u, and let all these edges belong to G_2. Again, G_2 has minimum degree at least D, and has at most $|U_2|D$ edges. Remove all the edges of G_2 and proceed to the next step.

Continuing in this manner until the number of remaining vertices of G becomes less than $\sqrt[r]{nd/4}$, we obtain a sequence of r-graphs $G_1, G_2, \ldots, G_p$. The construction ensures that properties (i),(ii),(iii) and (iv) hold. In addition, the edges of G that do not belong to any of the graphs G_i are either edges incident to one of the vertices deleted during the process when its degree was less than D (there are fewer than nD such edges), or edges that connect two of the remaining vertices when the number of vertices became smaller than $\sqrt[r]{nd/4}$ (there are fewer than $nd/r!4 \leq nd/8$ such edges). Since by definition $d > 4D$, property (v) holds as well, showing that there are indeed r-graphs G_i as claimed. $\square$

4 List coloring of graphs

In this section, we apply Lemma 3.1 for ordinary graphs (2-graphs). We prove the basic case $r = 2$ of Theorem 1.2 in the following form.

Theorem 4.1 *There exists a positive function $\delta(s) > 2^{-6s-11}/s^2$ such that for every $s \geq 12$ and every $d > 4\cdot(s4^{s+2})^2$, for any graph $G = (V, E)$ with n vertices and nd edges, there is an assignment of a list L_v of size s from a set of $2s - 1$ colors to each vertex $v \in V$ such that for every coloring assigning to each vertex a color from its list, the number of monochromatic edges is at least $\delta(s)\cdot dn$.*

Proof: Let $G = (V, E)$ be a graph with n vertices and nd edges. Apply to G Lemma 3.1 for $D = (s4^{s+2})^2$. Let $G_1, \ldots, G_p$ be the edge-disjoint subgraphs of G guaranteed by the lemma. For each i, $1 \leq i \leq p$, define $n_i = |U_i|$. Note that $D\sum_{i=1}^{p} n_i \geq nd/2$ implying that

$$\sum_{i=1}^{p} n_i \geq \frac{nd}{2D}. \tag{1}$$

For a subset B of vertices of G and an assignment of a list L_v of s colors from the set $S = \{1, 2, \ldots, 2s - 1\}$ to each vertex v of B, and for an integer i, $1 \leq i \leq p$, we say that a vertex u of G is *i-normal* if it belongs to $U_i - B$, the number of its neighbors in G_i that lie in $U_i \cap B$ is at least $0.5\sqrt{D}$, and for every subset $L \subset S$ of size $|L| = s$, there is at least one neighbor v of u in G_i that lies in B and whose list L_v is L. Let $T_i \subset U_i$ be the set of all i-normal vertices.

First, we claim that there is a subset B of vertices of G such that

(i) For each i, $|B \cap U_i| \leq \frac{2n_i}{\sqrt{D}}$, and

(ii) $\sum_{i=1}^{p} |T_i| > 0.9\sum_{i=1}^{p} n_i$.

Indeed, let B be a random set chosen by picking each vertex of G randomly and independently with probability $1/\sqrt{D}$. By the Chernoff bound (see [1, Appendix A1] or [14, p.26, (2.5)]) the probability that $|B \cap U_i| > \frac{2n_i}{\sqrt{D}}$ is at most $\exp\{-3n_i/8\sqrt{D}\}$. Since $n_i > \sqrt{nd}/2 \geq \sqrt{nD}$ for all i, this probability is at most $\exp\{-3\sqrt{n}/8\}$, and each G_i has at least $Dn_i/2 \geq D\sqrt{n}$ edges. So,

$$p \leq dn/D\sqrt{n} < n^{3/2}/D, \tag{2}$$

and hence the probability that (i) does not hold is at most

$$\exp\{-3\sqrt{n}/8\}\, p \leq \exp\{-3\sqrt{n}/8\}n^{3/2}/D.$$

For every fixed vertex $u \in U_i$, the probability that at least $\sqrt{D}/2$ vertices in B are neighbors of u in G_i is at least

$$1 - \exp\{-\sqrt{D}/8\} = 1 - \exp\{-2s4^s\} \geq 1 - \exp\{-4^s\}.$$

For $s \geq 12$ this is greater than 0.999. Let T'_i be the set of vertices of G_i that do not lie in B and have, in G_i, at least $\sqrt{D}/2$ neighbors in B. By linearity of expectation and by Markov's Inequality, with probability at least 1/2 we have

$$\sum_{i=1}^{p}(|T'_i| + |B \cap U_i|) \geq 0.99 \sum_{i=1}^{p} n_i, \tag{3}$$

so with positive probability both (3) and (i) hold. Fix a set B with these properties. By (3),

$$\sum_{i=1}^{p}|T'_i| \geq 0.99\sum_{i=1}^{p} n_i - \sum_{i=1}^{p}|B \cap U_i| \geq \sum_{i=1}^{p} n_i \left(0.99 - \frac{2}{\sqrt{D}}\right) \geq \sum_{i=1}^{p} n_i \left(0.99 - \frac{1}{4^{s+2}}\right) \geq 0.97 \sum_{i=1}^{p} n_i. \tag{4}$$

Assign to each vertex of B a list which is a random subset of size s of S. If a vertex u belongs to T'_i, then the probability that it is not i-normal is at most

$$\binom{2s-1}{s}\left(1 - \binom{2s-1}{s}^{-1}\right)^{\sqrt{D}/2} \leq 4^s\left(1 - 4^{-s}\right)^{8s4^s} \leq 4^s e^{-8s}.$$

For $s \geq 12$ this is less than 0.01, implying the existence of the required lists for the vertices of B by linearity of expectation (and the fact that $0.97 \cdot 0.99 > 0.9$).

Fix B and lists L_v for each $v \in B$ satisfying the two conditions above. To complete the proof, we show that if we now assign lists that are random subsets of S of size s to all other vertices of G, then with positive probability for every assignment of colors to the vertices from their lists, the number of monochromatic edges is at least

$$0.4 \cdot 4^{-s} \sum_{i=1}^{p} n_i \geq 0.4 \cdot 4^{-s}\frac{nd}{2D} = 0.4 \cdot 4^{-s}\frac{nd}{2s^2 4^{2s+4}} > \delta(s)dn.$$

For each i-normal vertex v, and for any assignment $c : B \cap U_i \mapsto S$ of colors to the vertices of $B \cap U_i$ from their lists, there are at least s colors that appear on the neighbors of v in G_i that lie in B. Fix an assignment f of colors to the vertices in $B \cap U_i$ from their lists. If v is i-normal and if we now assign a random list $L \subset S$ of size s to v, then the probability of the event $C(f, v, i)$ that L contains only colors assigned to the B-neighbors of v in G_i is at least $\binom{2s-1}{s}^{-1} > 4^{-s}$.

Let $I := \{i \in [p] : |T_i| > 0.5n_i\}$. By (ii) and the definition of I,

$$\frac{9}{10}\sum_{i=1}^{p} n_i < \sum_{i=1}^{p}|T_i| = \sum_{i \in I}|T_i| + \sum_{i \in [p]-I}|T_i| \leq \sum_{i \in I}|T_i| + \frac{1}{2}\sum_{i \in [p]-I} n_i \leq \sum_{i \in I}|T_i| + \frac{1}{2}\sum_{i=1}^{p} n_i - \frac{1}{2}\sum_{i \in I} n_i.$$

It follows that $0.4\sum_{i=1}^{p} n_i < \sum_{i \in I}|T_i| - \frac{1}{2}\sum_{i \in I} n_i \leq \frac{1}{2}\sum_{i \in I}|T_i|$ and hence

$$\sum_{i \in I}|T_i| \geq 0.8\sum_{i=1}^{p} n_i. \tag{5}$$

If $i \in I$, the probability of the event $C'(f, i)$ that $C(f, v, i)$ occurs for fewer than $0.5|T_i|4^{-s}$ vertices $v \in T_i$ is very small. Indeed, by the Chernoff bound (see [1, Appendix A1, Theorem A.1.13] or [14, p.26, (2.6)]), it is at most $e^{-|T_i|2^{-2s-3}}$. So, if $i \in I$, then the probability that $C'(f,i)$ occurs for at least one assignment f of colors from the lists to the vertices in $B \cap U_i$ is at most

$$e^{-|T_i|2^{-2s-3}} s^{|B\cap U_i|} \leq \exp\{\frac{2n_i}{\sqrt{D}} \ln s - 0.5n_i 2^{-2s-3}\} = \exp\{\frac{n_i}{4^{s+2}}\left(\frac{2\ln s}{s} - 1\right)\}$$

When $s \geq 12$, this is at most

$$\exp\{-0.5n_i 4^{-s-2}\} \leq \exp\{-0.5\sqrt{nd/4}\cdot 4^{-s-2}\} < \exp\{-\sqrt{n}(s4^{s+2})4^{-s-2}\} \leq e^{-12\sqrt{n}}.$$

So, by (2), the probability that $C'(f,i)$ occurs for at least one choice of i and f is at most

$$pe^{-12\sqrt{n}} \leq \frac{n^{3/2}}{D}e^{-12\sqrt{n}} \leq \frac{n^{3/2}}{(s4^{s+2})^2}e^{-12\sqrt{n}} \leq 1/2$$

for every n. It follows that with positive probability none of $C'(f,i)$ occurs at all. By (5) and (1), this gives the desired result. $\square$

5 Coloring simple hypergraphs

In this section we prove Theorem 1.2 for all r. We apply induction on r. Theorem 4.1 yields the base case $r = 2$ with $d_2(s) = 4\cdot(s4^{s+2})^2$, $R_2(s) = 2s-1$, and $\delta_2(s) = 2^{-6s-11}/s^2$. Let $r \geq 3$, $s \geq 6r$ and $d > d_r(s)$. Let G be a simple r-graph with n vertices and nd edges.

Remark 1. Let $k * G$ be the r-graph consisting of k vertex-disjoint copies of G. Then $k * G$ is a simple r-graph with the same average degree as G. Moreover, the statement of Theorem 1.2 holds for $k * G$ if and only if it holds for G. So, we may assume that n is large, and in particular,

$$\exp\{\sqrt[r]{n}\} \geq dn^4. \tag{6}$$

Put $M = 4s/\delta_{r-1}(s)$, $X = M^{2s+2}$ and $D = X^{r-1} = M^{(2s+2)(r-1)} \leq d^{1/2}$. Let $G_1 = (U_1, E_1), \ldots, G_p = (U_p, E_p)$ be the edge-disjoint r-graphs guaranteed by Lemma 3.1. Let $n_i = |U_i|$ for $i = 1, \ldots, p$. As in the previous section, we have $D\sum_{i=1}^p n_i \geq nd/2$ implying (1).

Choose randomly disjoint subsets $V_1, V_2, \ldots, V_M$ of V, where each vertex, randomly and independently, is chosen to lie in V_j with probability $\frac{2\ln M}{D^{1/(r-1)}} = \frac{2\ln M}{X}$. For each vertex $v \in V$, the *i-link of* v, denoted $l_i(v)$, is the set of all $(r-1)$-edges of the form $e - \{v\}$ for all edges $e \in E_i$ that contain v. Note that for each i, if $v \in U_i$, then $l_i(v)$ is an $(r-1)$-matching of size at least D (as G is simple). Say that a vertex $v \in U_i$ is *i-good* if $l_i(v)$ contains at least one edge in each V_j and call an i-good vertex *i-great* if it does not lie in $\cup_{j=1}^M V_j$. By the Chernoff bound (see [1, Appendix A1] or [14, p.26, (2.5)]), the probability of the event $B(G)$ that the size of at least one $V_j \cap U_i$ exceeds $\frac{2n_i(2\ln M)}{X}$ is at most $M\sum_{i=1}^p \exp\{-3n_i \ln M/4X\}$. Since $p < n^2$, by Lemma 3.1(ii) and (6),

$$\mathbf{P}(B(G)) \leq pM\exp\{-3\sqrt[r]{\frac{nd}{4}}\frac{\ln M}{4X}\} < n^2 M\exp\{-\sqrt[r]{n}X^{\frac{2r-2}{r}}\frac{\ln M}{4X}\} < n^2 M^{1-\sqrt[r]{n}X^{1/3}/4} < 10^{-2}. \tag{7}$$

The probability that a vertex $v \in U_i$ is not i-good is at most

$$M\left(1-\left(\frac{2\ln M}{X}\right)^{r-1}\right)^D \le M\exp\left\{-(2\ln M)^2\frac{D}{X^{r-1}}\right\} = M\exp\{-(2\ln M)^2\} < 0.001.$$

Hence with probability at least 0.9 there are at least $0.99\sum_{i=1}^p n_i$ vertices that are i-good for some i (counted with multiplicities). So by (7), with probability at least 1/2 this holds and $B(G)$ does not hold. Fix such sets V_j. With this choice, if T_i is the set of i-great vertices in G_i, then

$$\sum_{i=1}^p |T_i| \ge 0.99\sum_{i=1}^p n_i - \sum_{i=1}^p\sum_{j=1}^M |V_j\cap U_i| \ge \sum_{i=1}^p n_i\left(0.99 - M\frac{4\ln M}{X}\right) > 0.9\sum_{i=1}^p n_i. \tag{8}$$

Let $I := \{i\in[p] \,:\, |T_i| > 0.5n_i\}$. Similarly to deriving (5), (8) yields

$$\sum_{i\in I}|T_i| \ge 0.8\sum_{i=1}^p n_i. \tag{9}$$

Now, for each pair (v,i), where $i\in I$ and $v\in T_i$, keep in G_i exactly M edges containing v, one edge, call it $e(v,i,j)$, for each j, $1\le j\le M$, with $e(v,i,j)-\{v\}\subset V_j\cap U_i$, and omit all other r-edges from G_i. Let G'_i be the resulting hypergraph, and let $G' = \bigcup_{i\in I} G'_i$. For $j = 1,\dots,M$ and $i\in I$, let $G''_i(j)$ be the $(r-1)$-uniform hypergraph with $V(G''_i(j)) = V_j\cap U_i$ and $E(G''_i(j)) := \{e\cap V_j\cap U_i \,:\, e\in E(G'_i) \text{ and } |e\cap V_j| = r-1\}$. Let $G''(j) = \bigcup_{i\in I} G''_i(j)$ and $G''_i = \bigcup_{j=1}^M G''_i(j)$. Let $G'' := \bigcup_{j=1}^M G''(j) = \bigcup_{i\in I} G''_i$. Then $|E(G''_i(j))| = |T_i| > 0.5n_i$ for each $i\in I$ and $j\in\{1,\dots,M\}$, and by (9) and (1) for every $j = 1,\dots,M$,

$$|E(G''(j))| = \sum_{i\in I}|E(G''_i(j))| = \sum_{i\in I}|T_i| \ge 0.8\cdot\sum_{i=1}^p n_i \ge \frac{2nd}{5D}. \tag{10}$$

Since $|V_j\cap U_1| \le \frac{4n_1\ln M}{X}$,

$$|E(G''(j))|/|V(G''(j))| \ge \frac{2nd}{5D}\frac{X}{4n\ln M} = \frac{dX}{10D\ln M}.$$

So, by the induction assumption, there is an assignment L_j of lists of size s from the set $\{(j-1)R_{r-1}(s)+1, (j-1)R_{r-1}(s)+2,\dots, jR_{r-1}(s)\}$ to the vertices in $V_j\cap U_1$ such that for every coloring of $V_j\cap U_1$ from these lists, the number of monochromatic edges in $G''(j)$ is at least

$$|E(G''(j))|\delta_{r-1}(s) = \sum_{i\in I}|T_i|\delta_{r-1}(s) \ge \frac{2nd}{5D}\delta_{r-1}(s). \tag{11}$$

Fix such assignments $L_1,\dots,L_m$. Let F denote the set of all colorings of vertices in $V(G'')$ from these lists. For every $f\in F$, let $I(f)$ be the set of indices $i\in I$ such that the number of monochromatic edges under f in G''_i is at least $0.5\delta_{r-1}(s)|E(G''_i)| = 0.5\delta_{r-1}(s)M|T_i| = 2s|T_i|$.

To every vertex $v\in U_1 - V(G'')$ we assign a list chosen at random among all subsets of size s of $\{1,2,\dots,MR_{r-1}(s)\}$ uniformly and independently from all other vertices. We will show that with positive probability this choice will satisfy the statement of our theorem.

Consider a coloring $f \in F$ and $i \in I(f)$. Say that a vertex $v \in T_i$ is (i,f)-*dangerous* if at least s $(r-1)$-edges from its link in G'_i are monochromatic in G''_i under f. Let $x_i(f)$ denote the number of (i,f)-dangerous vertices in T_i. Since $i \in I(f)$, we have

$$x_i(f) \geq (|E(G''_i)| - s|T_i|)/M \geq |E(G''_i)|/2M \geq s|T_i|/M. \tag{12}$$

With our choice of lists, the probability that an (i,f)-dangerous vertex will get a list consisting only of s colors that appear as the color of a monochromatic edge in its link in G''_i is at least $\binom{MR_{r-1}(s)}{s}^{-1} \geq M^{-2s}$. Hence the probability of the event $C_i(f)$ that this occurs for less that $0.5M^{-2s}\frac{|E(G''_i)|}{2M}$ times is at most

$$\exp\{-M^{-2s}\frac{|E(G''_i)|}{16M}\} \leq \exp\{-M^{-2s}\frac{s|T_i|}{8M}\} \leq \exp\{-0.5sM^{-2s-1}\frac{n_i}{8}\} \leq \exp\{-M^{-2s-1}n_i\}.$$

Since $|V_j \cap U_i| \leq \frac{4n_i \ln M}{X}$, the probability of the event C_i that at least one of $C_i(f)$ occurs is at most

$$s^{4n_i \ln M/X} \exp\{-M^{-2s-1}n_i\} = \exp\left\{-\frac{n_i}{X}\left(\frac{X}{M^{2s+1}} - 4\ln s \ln M\right)\right\} \leq \exp\left\{-\frac{n_i}{X}\right\}.$$

Thus the probability of the event C that at least one of C_i occurs is at most $p\exp\{-\frac{n_p}{X}\}$. Recal that $p < n^2$ and (by Lemma 3.1(ii)) $n_p \geq \sqrt[r]{nd/4}$. So by (6), $p\exp\{-\frac{n_p}{X}\} < 1$. Thus, there exists a list assignment L such that none of $C_i(f)$ occurs.

Let $\widetilde{f}$ be any L-coloring of G and $f \in F$ be its restriction to $V(G'')$. We will prove that there are at least $\frac{nd}{DX}$ monochromatic edges in $\widetilde{f}$. By (11), for the number Z of monochromatic edges in coloring f of G'' we have

$$Z \geq M\sum_{i\in I}|T_i|\delta_{r-1}(s) \geq M\frac{2nd}{5D}\delta_{r-1}(s) \geq \frac{8snd}{5D}. \tag{13}$$

Suppose that G''_i has z_i of these monochromatic edges. By the definition of $I(f)$,

$$\sum_{i\in I-I(f)} z_i < \sum_{i\in I-I(f)} 0.5\delta_{r-1}(s)M|T_i| \leq \sum_{i\in I} 0.5\delta_{r-1}(s)M|T_i| \leq Z/2. \tag{14}$$

By (12), $x_i(f) \geq z_i/2M$ for each $i \in I(f)$. Since $C_i(f)$ does not occur, the number of monochromatic edges in G_i under $\widetilde{f}$ is at least $0.5M^{-2s}x_i(f)$. So by (14) and (13), the total number of monochromatic edges in G is at least

$$\sum_{i\in I(f)} 0.5M^{-2s}x_i(f) \geq \sum_{i\in I(f)} 0.5M^{-2s}\frac{z_i}{2M} \geq \frac{1}{4}M^{-2s-1}\frac{Z}{2} \geq M^{-2s-1}\frac{nd}{D} \geq \frac{nd}{DX}. \quad \square$$

6 A geometric application

In this section, we prove Theorem 1.3. For convenience, we restate it here.

Theorem 6.1 *For any finite set X in the Euclidean plane and for any positive integer s, there is an assignment of a list of size s to every point of the plane, such that whenever we color the points of the plane from their lists, there is a monochromatic isometric copy of X.*

Proof: Put $r = |X|$. By Theorem 1.1, it suffices to show that for any d there is a d-regular simple r-uniform hypergraph whose set of vertices is a finite set of points in the plane, such that the vertices of each edge form an isometric copy of X.

Let $\{v_{11}, v_{12}, \ldots, v_{1r}\}$ be r points in R^2 that form a copy of X. For each i, $2 \leq i \leq d$, let $\{v_{i1}, v_{i2}, \ldots, v_{ir}\}$ be a set obtained from X by applying to it a rotation. The rotations are chosen in a generic manner, to ensure that no difference between two vectors of one copy is equal to a difference between two vectors of another copy. The set of vertices of our hypergraph is the following set of r^d sums:

$$\{v_{1j_1} + v_{2j_2} + v_{3j_3} + \cdots + v_{dj_d} : \ 1 \leq j_q \leq r \text{ for all } \ q\}$$

and the set of edges is the set of all dr^{d-1} r-tuples

$$\{v_{1j_1} + v_{2j_2} + v_{3j_3} + \cdots + v_{dj_d} : \ 1 \leq j_q \leq r\}.$$

Note that in each such edge, all summands but the qth are fixed, hence every edge forms an isometric copy of X.

It is not difficult to check that the above hypergraph is simple, r-uniform and d-regular, and the result thus follows from Theorem 1.1. $\Box$

7 Concluding remarks and open problems

- By inspecting the proof of the main result one can check that the estimates it provides for $d_r(s)$ and $\delta_r(s)$ satisfy

$$\delta_r(s) \geq 2^{-3^r r! s^{r-1}} \quad \text{and} \quad d_r(s) \leq 2^{3^{r+1} r! s^{r-1}}.$$

  In particular, for any fixed r, $d_r(s) \leq 2^{O(s^{r-1})}$. It is not difficult to check that $d_r(s) \geq r^{\Omega(s)}$, and it seems plausible to conjecture that for any fixed r, $d_r(s) = 2^{O(s)}$. This holds for $r = 2$ by the results in [3] and [10] (or more generally, [4]), but remains open for any fixed $r \geq 3$.

- As mentioned in the introduction, our main result provides a linear time algorithm for computing, for a given input simple r-graph, a number s such that its list chromatic number is at least s and at most $f(s)$ for some explicit function f. The same result implies that if a simple r-graph is s-list colorable, then there is a linear time algorithm for finding a proper list coloring of it from lists of size $f(s)$. There is no such known result for ordinary coloring, even for $r = 2$. Indeed, the minimum number $Q = Q(3, n)$ for which there is a polynomial time algorithm that colors properly a given 3-colorable graph on n vertices by Q colors, is about $\Theta(n^{0.207})$. This is shown in [5], improving several earlier results. It is known that if $P \neq NP$ then there is no polynomial time algorithm that finds a proper 4-coloring of a 3-colorable graph (see [17], [11]), and there is no polynomial time algorithm that finds a proper $q^{(\log q)/25}$-coloring of a q-colorable graph, as shown in [18]. Moreover, under some plausible hardness assumptions in Complexity Theory, there is no polynomial time algorithm that produces a proper Q-coloring of a q colorable graph for any fixed $3 \leq q \leq Q$, as shown in [6].

- The analog of Theorem 6.1 holds for R^m with any $m \geq 2$ (and in fact for $m \geq 3$ it is enough to only allow translations and rotations in one direction). An appropriate version works for any infinite group acting on a set.

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