Coloring graphs with sparse neighborhoods

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Abstract

It is shown that the chromatic number of any graph with maximum degree d in which the number of edges in the induced subgraph on the set of all neighbors of any vertex does not exceed d^2/f is at most $O(d/\log f)$. This is tight (up to a constant factor) for all admissible values of d and f.

1 Introduction

The chromatic number $\chi(G)$ of a graph G is the minimum number of colors required to color all its vertices so that adjacent vertices get distinct colors. It is easy and well known that if d is the maximum degree of G then $\chi(G) \leq d+1$. This upper bound can be improved if the graph has sparse neighborhoods, namely, if no subgraph on the set of all neighbors of a vertex spans too many edges. The first instance of a result of this type is Brooks' Theorem [5], which asserts that if no neighborhood contains $\binom{d}{2}$ edges (that is, if G contains no copy of the complete graph on $d+1$ vertices), then $\chi(G) \leq d$. Molloy and Reed [13] proved that for every $\epsilon > 0$ there is some $\delta > 0$ such that if no neighborhood contains more than $(1-\epsilon)\binom{d}{2}$ edges, then $\chi(G) \leq (1-\delta)(d+1)$. Johansson [7] proved that if each neighborhood contains no edges at all (that is, if G is triangle-free), then $\chi(G) \leq O(d/\log d)$. Related results for independence numbers of graphs with sparse neighborhoods appeared earlier in [1]. Our main result in the present note is the following.

Theorem 1.1 *There exists an absolute positive constant c such that the following holds. Let $G = (V,E)$ be a graph on n vertices with maximum degree d in which the neighborhood $N(v)$ of any vertex $v \in V$ spans at most d^2/f edges. Then the chromatic number of G is at most $c\frac{d}{\log f}$.*

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This result supplies an interpolation between the above mentioned bounds and is tight, up to the multiplicative constant c, for all admissible values of d and f, as stated in the following result.

Proposition 1.2 *There exists an absolute positive constant b such that the following holds. For every positive integer d and every f satisfying $2 \le f \le 2d^2$, there is a graph G with maximum degree at most d in which the neighborhood of any vertex spans at most d^2/f edges and $\chi(G) \ge bd/\log f$.*

The proof of the main result is based on probabilistic arguments, and is described in the following section. The final section contains some concluding remarks.

2 The proof

We first prove Theorem 1.1 for $f \ge d^{4\epsilon}$, where $\epsilon > 0$ is some fixed constant (say, $\epsilon = 1/28$.) We make no attempt to optimize our absolute constants here and in the rest of the paper.

Theorem 2.1 *Let $G = (V, E)$ be a graph on n vertices with maximum degree d in which the neighborhood $N(v)$ of any vertex $v \in V$ spans at most $d^{2-4\epsilon}$ edges for some fixed $\epsilon > 0$. Then the chromatic number of G is at most $O(d/(\epsilon^3 \log d))$.*

Our proof relies heavily on the following result of Johansson [7] mentioned in the introduction.

Proposition 2.2 ([7]) *If G is a triangle-free graph with maximum degree d then*

$$\chi(G) \le O(\frac{d}{\log d}). \qquad \Box$$

We need the following lemma.

Lemma 2.3 *Let $\epsilon > 0$ be a fixed real. Let $G = (V, E)$ be a graph on n vertices with maximum degree d in which the neighborhood $N(v)$ of each vertex $v \in V$ spans at most $d^{2-4\epsilon}$ edges. Then there exists a partition of the vertex set $V = V_1 \cup \ldots \cup V_k$ with $k = \Theta(d^{1-\epsilon}/\epsilon^2)$, such that for any $1 \le i \le k$ the induced subgraph $G[V_i]$ is triangle-free and has maximum degree at most $O(d/\epsilon^2 k) = O(d^\epsilon)$.*

Proof. We may and will assume, whenever this is needed, that d is sufficiently large. First partition the vertices of G into $d^{1-\epsilon}$ parts $U_i, 1 \le i \le d^{1-\epsilon}$. To do so color the vertices of the graph randomly and independently by $d^{1-\epsilon}$ colors. For a vertex $v \in V(G)$, and a vertex u adjacent to it, call u a *bad neighbor* of v, if they have at least $d^{1-2\epsilon}$ common neighbors. Otherwise call u a *good neighbor* of v. Since the neighborhood of any vertex spans at most $d^{2-4\epsilon}$ edges, there are at most $2d^{1-2\epsilon}$ bad neighbors for any vertex. For any vertex v define three types of "bad" events. Let A_v be the event that v has more than $2d^\epsilon$ neighbors of the same color as v. Let B_v be the event that v has more than $10/\epsilon$ bad neighbors with the same color as v. Finally, let C_v be the event that the set of good neighbors of v which have the same color as v spans more than $100/\epsilon^2$ edges. We use the symmetric version of the Lovász Local Lemma (see e.g. [4]) to prove that with positive probability no bad events happen.

Note that each of the events A_v, B_v and C_v is independent of all but at most $O(d^2)$ others, as it is independent of all events A_u, B_u, C_u corresponding to vertices u whose distance from v is bigger than 2. Since the degree of any vertex in its color class is binomially distributed with mean at most d^ϵ, the standard Chernoff estimates (see, e.g., [4] , Appendix A) imply that the probability that the vertex v has more than $2d^\epsilon$ neighbors of the same color as that of v is at most $e^{-\Omega(d^\epsilon)} < d^{-3}$. Clearly

$$Pr[B_v] \leq \binom{2d^{1-2\epsilon}}{\frac{10}{\epsilon}} (\frac{1}{d^{1-\epsilon}})^{\frac{10}{\epsilon}} < d^{-3}.$$

To bound the probability of C_v we need the following simple observation. If a graph has more than $100/\epsilon^2$ edges than by Vizing's theorem [14] it either has a vertex of degree at least $9/\epsilon$ or a matching with at least $9/\epsilon$ edges. Therefore C_v can happen only if we have a matching of size at least $9/\epsilon$ on the good neighbors of v which have the same color as v or there is a good neighbor u of v, such that u and v have at least $9/\epsilon$ common neighbors with the same color as v . The probability of the first event is bounded by

$$\binom{d^{2-4\epsilon}}{\frac{9}{\epsilon}} ((d^{1-\epsilon})^{-2})^{\frac{9}{\epsilon}} < 0.5d^{-3},$$

and the probability of the last one is at most

$$d \binom{d^{1-2\epsilon}}{\frac{9}{\epsilon}} (\frac{1}{d^{1-\epsilon}})^{\frac{9}{\epsilon}} < 0.5d^{-3}.$$

Therefore the probability of C_v is at most d^{-3}. Thus, by the Local Lemma, with positive probability none of the events A_v, B_v or C_v happen. Thus we have a partition $U_i, 1 \leq i \leq d^{1-\epsilon}$ such that in each induced subgraph $G[U_i]$ the maximum degree is at most $2d^\epsilon$ and the neighborhood of any vertex v in it has the property that one can remove from it a set $S_{v,i}$ of at most $100/\epsilon^2 + 10/\epsilon \leq 110/\epsilon^2$ vertices such that the remaining ones span no edges. Fix such a partition.

Recall that a graph is p-degenerate if any subgraph of it contains a vertex of degree at most p. Such graphs are trivially $p+1$ colorable. Construct an auxiliary digraph D_i on the set of vertices U_i as follows. For each vertex $v \in U_i$ the edges from v are exactly the ordered pairs $\{(v,u) | u \in S_{v,i}\}$. By definition, the digraph D_i has maximum outdegree at most $110/\epsilon^2$ and therefore it is $220/\epsilon^2$ degenerate. Thus D_i can be properly colored by at most $220/\epsilon^2 + 1$ colors. Note that each color class is now a triangle-free graph. Altogether this gives a partition of the graph into at most $O(d^{1-\epsilon}/\epsilon^2)$ parts such that each part induces a triangle-free graph with maximum degree at most $2d^\epsilon$. This completes the proof. □

Now we are ready to prove the theorem.

Proof of Theorem 2.1. Let $V_1, \ldots, V_k$ be a partition of the vertices of G satisfying the properties in Lemma 2.3 . Then by Johansson's result (Proposition 2.2) the chromatic number of each induced subgraph $G[V_i]$ is at most $O(\frac{d}{\epsilon^2 k}/\log(\frac{d}{\epsilon^2 k}))$. Coloring each subgraph $G[V_i]$ by its own colors we get a proper coloring of G. Therefore

$$\chi(G) \leq k\ O\left(\frac{d}{\epsilon^2 k}/\log(\frac{d}{\epsilon^2 k})\right) = O\left(\frac{d}{\epsilon^3 \log d}\right).$$

□

An immediate consequence of the last theorem is the following:

Corollary 2.4 *Let $H=(V,E)$ be a fixed graph so that $H-v$ is bipartite for some $v \in V$. Then the chromatic number of any graph G with maximum degree d, which does not contain a copy of H is at most $O(d/\log d)$. In particular, this is true for $H = C_k$, the cycle of length k.*

Proof. We may and will assume, that d is sufficiently large as a function of the size of H. For each vertex u of G the induced subgraph $G[N(u)]$ on the neighborhood of u contains no copy of the bipartite graph $H-v$. Thus by the known results on Zarankiewicz's problem (see, e.g., [11], Problem 10.37) the number of edges in $G[N(v)]$ is at most $d^{2-4\epsilon}$ for some fixed $\epsilon = \epsilon(H)$. Now use Theorem 2.1 to color G. □

The general case of Theorem 1.1 cannot be proved using the arguments in the proof of Theorem 2.1 since if f is much smaller than the degree d the desired partition cannot be obtained in one step since this would cause more dependencies than we may allow. To overcome this difficulty we apply an approach similar to the one used in [2] and construct the desired partition by a sequence of random halving steps. This is done using the following lemma.

Lemma 2.5 *Let $G=(V,E)$ be a graph on n vertices with maximum degree $d \geq 2$ in which the neighborhood $N(v)$ of any vertex $v \in V$ spans at most s edges. Then there exists a partition of V into two subsets $V = V_1 \cup V_2$ such that the induced subgraph $G[V_i], i=1,2$, has maximum degree at most $d/2 + 2\sqrt{d \log d}$ and the neighborhood $N(u)$ of any vertex u in $G[V_i], i=1,2$ spans at most $s/4 + 2d^{3/2}\sqrt{\log d}$ edges.*

Proof. Partition the set of vertices V into two subsets V_1 and V_2 by choosing for each vertex randomly and independently an index i to be either 1 or 2 and placing it into V_i. For $i=1,2$ let G_i be the induced subgraph of G on V_i. For each $v \in V$ let A_v be the event that the degree of v in G_i is greater than $d/2 + 2\sqrt{d \log d}$ and let B_v be the event that the neighborhood of v in G_i spans more than $s/4 + 2d^{3/2}\sqrt{\log d}$ edges. Observe that if none of the events A_v, B_v holds, then our partition satisfies the assertion of the lemma. Hence it suffices to show that with positive probability no event A_v, B_v happens. We prove this by applying the Local Lemma. Since the number of neighbors of any vertex v in $G_i, i=1,2$, is a binomially distributed random variable with parameters $d(v) \leq d$ and $1/2$, it follows by the standard estimates for Binomial distributions (cf. , e.g., [4], Appendix A) that for every $v \in V$

$$Pr(A_v) \leq 2e^{-\frac{2(2\sqrt{d \log d})^2}{d}} \leq 2d^{-8}.$$

To bound the probability of the event B_v we use a large deviation inequality for martingales. Denote by X_v^i the number of edges spanned by the neighborhood of v in G_i. Then the expected value of X_v^i satisfies $EX_v^i \le s/4$. Observe that by shifting a neighbor u of v from G_1 to G_2 or vice versa the value of X_v^i can change by at most $d(u) \le d$. Since the number of neighbors of v is at most d, the large deviation inequality for vertex exposure martingale (see, e.g., [12], p. 149) implies that

$$Pr(X_v^i - EX_v^i \ge t) \le e^{-\frac{2t^2}{d \cdot d^2}}.$$

By taking $t = 2d^{3/2}\sqrt{\log d}$ we obtain

$$Pr(B_v) \le e^{-2\frac{(2d^{3/2}\sqrt{\log d})^2}{d^3}} < 2d^{-8}.$$

Clearly each event A_v, B_v is independent of all but at most $2d^2$ others, as it is independent of all events A_u, B_u corresponding to vertices u whose distance from v is bigger than 2. Since $e \cdot 2d^{-8} \cdot (2d^2+1) < 1$ we conclude, by the Local Lemma, that with positive probability no event A_v, B_v holds. This completes the proof of the lemma. □

Proof of Theorem 1.1. Let $G = (V, E)$ be a graph with maximum degree d in which the neighborhood $N(v)$ of any vertex $v \in V$ spans at most d^2/f edges. Since the chromatic number of G is at most $d+1$ we may and will assume, whenever this is needed, that f is sufficiently large. First we consider the case $f > d^{1/7}$. In this case the result of the theorem follows from Theorem 2.1 with $\epsilon = 1/28$. Hence we can assume that $f \le d^{1/7}$. Apply Lemma 2.5 to split G into two induced subgraphs $G[V_i], i = 1, 2$ such that the maximum degree in each $G[V_i]$ is at most $d/2 + 2\sqrt{d \log d}$ and the neighborhood $N(u)$ of any vertex u in $G[V_i], i = 1, 2$, spans at most $d^2/(4f) + 2d^{3/2}\sqrt{\log d}$ edges. By applying Lemma 2.5 again to each of these two graphs we obtain a splitting of G into four induced subgraphs. Continuing in this manner we obtain, after j such halving steps, a partition of G into 2^j induced subgraphs. Define two sequences d_t and $s_t, 0 \le t \le j$ as follows, $d_0 = d$, $s_0 = d^2/f$ and for all $t < j$:

$$d_{t+1} = d_t/2 + 2\sqrt{d_t \log d_t}, \quad s_{t+1} = s_t/4 + 2d_t^{3/2}\sqrt{\log d_t}$$

Note that by Lemma 2.5, d_t is an upper bound for the maximum degree and s_t is an upper bound for the number of edges spanned by the neighborhood of any vertex in any of the 2^t induced subgraphs of G obtained after t halving steps.

Let j be the smallest integer for which $(8d/2^j)^{1/7}$ becomes less than f. We claim that $d_j \le 8d/(2^j)$ and $s_j \le (8d/2^j)^2/f$. Thus we can apply Theorem 2.1 (with $\epsilon = 1/28$) to each of the 2^j induced subgraphs obtained after the j-th step of the partition. This implies that the chromatic number of each such induced subgraph is at most

$$O\left(\frac{8d/2^j}{\log(8d/2^j)}\right) = O\left(\frac{d}{2^j \log f}\right).$$

By coloring each of the 2^j induced subgraphs with new colors we obtain a proper coloring of G by $O(d/\log f)$ colors. In order to complete the proof of the theorem it thus remains to show that $d_j \leq 8d/(2^j)$ and $s_j \leq (8d/2^j)^2/f$.

Clearly, $d_t \geq d/(2^t) > f^7/8$ is large enough by our assumption, hence $d_{t+1} \leq d_t/2 + d_t^{2/3} \leq \frac{1}{2}(d_t^{1/3}+1)^3$ for all $t < j$. Thus by taking cubic roots and subtracting $\frac{1}{2^{1/3}-1}$ from both sides we obtain

$$d_{t+1}^{1/3} - \frac{1}{2^{1/3}-1} \leq \frac{1}{2^{1/3}}(d_t^{1/3}+1) - \frac{1}{2^{1/3}-1} = \frac{1}{2^{1/3}}\left(d_t^{1/3} - \frac{1}{2^{1/3}-1}\right).$$

Therefore

$$d_j^{1/3} - \frac{1}{2^{1/3}-1} \leq \frac{1}{2^{j/3}}\left(d_0^{1/3} - \frac{1}{2^{1/3}-1}\right),$$

and, since $d_0 = d$ and $2^{1/3} - 1 > 1/4$,

$$d_j^{1/3} \leq \frac{d^{1/3}}{2^{j/3}} + 4 \leq 2\frac{d^{1/3}}{2^{j/3}}.$$

The last inequality follows from the assumption that $d/2^{j-1} > f^7/8$ is large enough. Thus $d_j \leq 8d/(2^j)$. Note that the same proof also shows that $d_t \leq 8d/2^t$ for all $t \leq j$. Since by definition

$$s_t \geq \frac{s_0}{4^t} = \frac{d^2}{4^t f} = \frac{1}{64}\frac{(\frac{8d}{2^t})^2}{f} \geq \frac{1}{64}\left(\frac{8d}{2^t}\right)^{2-1/7} \geq \frac{1}{64}d_t^{2-1/7}$$

for all $t < j$, we obtain that $s_{t+1} \leq s_t/4 + 3s_t^{5/6} \leq \frac{1}{4}(s_t^{1/6}+2)^6$. Hence by taking sixth roots and subtracting $\frac{2}{4^{1/6}-1}$ from both sides we obtain

$$s_{t+1}^{1/6} - \frac{2}{4^{1/6}-1} \leq \frac{1}{4^{1/6}}(s_t^{1/6}+2) - \frac{2}{4^{1/6}-1} = \frac{1}{4^{1/6}}\left(s_t^{1/6} - \frac{2}{4^{1/6}-1}\right).$$

Therefore

$$s_j^{1/6} - \frac{2}{4^{1/6}-1} \leq \frac{1}{4^{j/6}}\left(s_0^{1/6} - \frac{2}{4^{1/6}-1}\right),$$

and, since $s_0 = d^2/f$ and $4^{1/6} - 1 > 1/4$,

$$s_j^{1/6} \leq \frac{d^{2/6}}{4^{j/6}f^{1/6}} + 8 \leq 2\frac{d^{2/6}}{4^{j/6}f^{1/6}}.$$

Thus $s_j \leq (8d/2^j)^2/f$, completing the proof of the theorem. □

We conclude the section with the (simple) proof of Proposition 1.2, which shows that the assertion of Theorem 1.1 is tight.

Proof of Proposition 1.2. It is well known (see, e.g., [10], [6]) that there is an absolute constant C so that for every integer Δ there is a triangle-free graph G on m vertices with maximum degree Δ containing no independent set of size at least $Cm\log\Delta/\Delta$. (In fact the above holds even for graphs with arbitrarily large girth; for large Δ the best known estimate is $(2+o(1))m\log\Delta/\Delta$, where the logarithm is in base e.) The chromatic number of each such graph is clearly at least $\Delta/(C\log\Delta)$

and as it is triangle-free it has no edges at all in any neighborhood. Therefore, taking $\Delta = d$, the assertion of Proposition 1.2 for any $f \geq d^{\Omega(1)}$ follows. We thus may assume that, say, f is an integer satisfying $2 \leq f < d/4$. In this case let H be a triangle-free graph with maximum degree $\Delta = 2f$, m vertices, and no independent set of size $Cm\log(2f)/(2f)$. Let G be the graph obtained from H by replacing each vertex of H by a clique on $\lfloor d/(2f+1)\rfloor$ vertices. Then the maximum degree of G is smaller than d, it has $m\lfloor d/(2f+1)\rfloor$ vertices, it contains no independent set of size at least $Cm\log(2f)/(2f)$, and the maximum number of edges in a neighborhood of a vertex in it is smaller than $3d^2/(4f) \leq d^2/f$. It follows that the chromatic number of G is at least

$$\frac{m\lfloor d/(2f+1)\rfloor}{Cm\log(2f)/(2f)} = \Omega(d/\log f),$$

completing the proof. $\square$

3 Concluding remarks

- Some of the results described here can be extended to list colorings. The *choice number* $ch(G)$ of a graph $G = (V, E)$ is the smallest integer k such that for every assignment of a list $S(v)$ of k colors to each vertex v of G there is a proper coloring of G assigning to each vertex a color from its list. Clearly $ch(G) \geq \chi(G)$ for every graph G and it is well known that the inequality may be strict. It is not difficult to extend some of the results of this note to choice numbers. Thus, for example, using the technique of [3] it is not difficult to show that if G satisfies the assumptions of Theorem 2.1 and $d \geq (\log n \log\log n)^{1/\epsilon}$ then the inequality $ch(G) \leq O(d/(\epsilon^3 \log d))$ (which is stronger than the assertion of the theorem) holds as well.

  Indeed, as proved by Johansson in [7], the choice number of any triangle-free graph with maximum degree d is at most $O(d/\log d)$. Given lists of colors L_v of size $O(\frac{d}{\epsilon^3 \log d})$ for each vertex v, partition the set of all colors $X = \cup_{v\in V} L_v$ into k sets $X_1, \ldots, X_k$ by choosing for each color randomly and independently an index i between 1 and k and by placing it in X_i. Since for all vertices $v \in V$ the random variable $|L_v \cap X_i|$ is binomially distributed with parameters $O(\frac{d}{\epsilon^3 \log d})$ and $1/k$, then by a standard large deviation inequality (cf. , e.g., [4], Appendix A)

$$Pr\left(|L_v \cap X_i| \leq O(\frac{d}{\epsilon^3 k \log d})\right) < e^{-\Omega(\frac{d}{\epsilon^3 k \log d})} = e^{-\Omega(\frac{d^\epsilon}{\epsilon \log d})} < \frac{1}{n^2}.$$

  Therefore with positive probability no such event happens. This implies that there exists a partition of the colors into k pairwise disjoint parts with the property that $|L_v \cap X_i| \geq O(\frac{d}{\epsilon^3 k \log d})$ for all i and $v \in V$. Take one such partition. Since the induced subgraph $G[V_i]$ constructed in the proof of Theorem 2.1 is triangle-free and has maximum degree $O(d/\epsilon^2 k)$, Johansson's result implies that its choice number is at most $ch(G[V_i]) \leq O(\frac{d}{\epsilon^2 k}/\log(\frac{d}{\epsilon^2 k})) = O(\frac{d}{\epsilon^3 k \log d})$. Therefore one can color the induced subgraph $G[V_i]$ using only colors from X_i. Since the sets

X_i are pairwise disjoint this gives a proper coloring of the vertices of the graph G using the original lists of colors.

Similarly, if G satisfies the assumptions of Corollary 2.4 and $d \geq (\log|V| \log\log|V|)^{1/\epsilon}$ then the inequality $ch(G) \leq O(d/\log d)$ holds.

- It is worth noting that there is no analog of Theorem 1.1 if the assumption that the maximum degree of G is d is replaced by the assumption that G is d-degenerate. A graph is d-degenerate if any induced subgraph of it contains a vertex of degree at most d. It is easy and well known that each such graph is $(d+1)$-colorable (see, e.g., [9], page 8). However, it turns out that for every d there is a triangle-free d-degenerate graph G_d whose chromatic number is $d+1$. This is trivially true for $d=1$. Assuming it holds for $d-1$, we prove it for d, $d \geq 2$. Let G_d consist of d pairwise vertex disjoint copies $H_1, H_2, \ldots, H_d$ of G_{d-1} together with an independent set X of size m^d, where m is the number of vertices of G_{d-1}. For each choice of d vertices, one from each H_i, there is a unique vertex of X joined to all these vertices. It is easy to check that the resulting graph is a triangle-free, d-degenerate graph of chromatic number $d+1$.

- Our main result here determines the asymptotic behavior of the maximum possible chromatic number of graphs with a given maximum degree and a given bound on the maximum number of edges in a neighborhood. A related conjecture is the following one, dealing with the maximum possible chromatic number of graphs with forbidden subgraphs.

  Conjecture 3.1 *For every fixed graph H there exists a positive constant c_H such that the chromatic number of any graph G with maximum degree d that contains no copy of H is at most $c_H d/\log d$.*

  Some version of this conjecture is suggested in [8], where it is shown that it almost holds: the chromatic number of any graph G as above is at most $c_H d \log\log d/\log d$. Note that Johansson's result [7] stated as Proposition 2.2 here shows that the conjecture holds for $H = K_3$, and Corollary 2.4 shows it holds for all graphs H containing a vertex v such that $H - v$ is bipartite.

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