Correction: Basic Network Creation Games

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Abstract

We prove a previously stated but incorrectly proved theorem: there is a diameter-3 graph in which replacing any edge $\{v, w\}$ of the graph with $\{v, w'\}$, for any vertex w', does not decrease the total sum of distances from v to all other nodes (a property called *sum equilibrium*).

Theorem 5 in [1] states that there exists a diameter-3 sum equilibrium graph, that is, an undirected graph such that, for every edge $\{v, w\}$ and every node w', replacing edge $\{v, w\}$ with $\{v, w'\}$ does not decrease the total sum of distances from v to all other nodes (and thus no vertex v has incentive to swap an incident edge). In this short note, we observe an error in the original construction and proof, but present a different example that is indeed a diameter-3 sum equilibrium graph, thereby restoring the theorem.

First we describe why Figure 3 of [1] is not in sum equilibrium. Specifically, vertex d_1 has an incentive to replace the edge $\{d_1, c_{1,1}\}$ with $\{d_1, c_{2,1}\}$, as the total distance is 27 in the first case and 26 in the last. The original proof ignores that $c_{2,1}$ is a neighbor of $c_{1,1}$ and, hence, Lemma 8 of [1] implies that the distance from d_1 to $c_{1,1}$ increases by 1 and not by 2 as claimed.

Figure 1 below presents a diameter-3 sum equilibrium graph G (which is also simpler than the original construction). In this instance, vertices v_2, v_4, v_5, and v_7 have local diameter 2 so, by Lemma 6 of [1], they have no incentive to swap any edge. (Lemma 6 states that a vertex of local diameter 2 never has incentive to swap an incident edge, as the number of distance-1 neighbors remains fixed, and thus the number of nodes at distance ≥ 2 remains fixed, so keeping their distances equal to 2 is optimal.) Among the remaining vertices, by symmetry, it suffices to prove that v_1 and v_3 do not have an incentive to swap edges.

Consider vertex v_i for $i \in \{1, 3\}$. Let G_{-i} be the graph obtained by removing vertex v_i and its incident edges; refer to Figure 2. The sum of distances for v_i in G is 13. Because v_i has degree 2, the smallest possible sum of distances for v_i is 12, which can be obtained if v_i were connected to two vertices that form a dominating set in G_{-i}. (A dominating set of cardinality larger than

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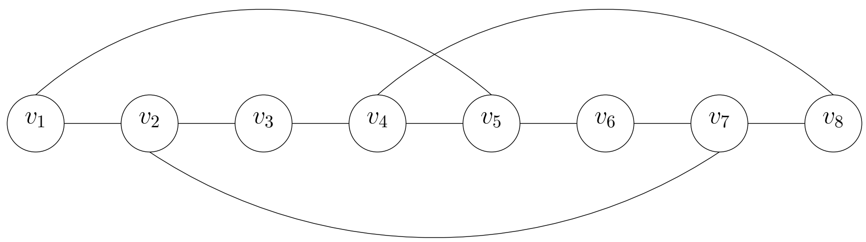

Figure 1: A diameter-3 sum equilibrium graph.

two can safely be ignored because v_i, having degree 2, cannot connect to all its vertices in order to reduce the sum of distances to less than 13.) Furthermore, the only dominating set in G_{-i} with cardinality 2 consists of vertices with degree 3 in G_{-i}, i.e., vertices v_4 and v_7 for G_{-1} and vertices v_5 and v_7 for G_{-3}. (To see that, note that, because G_{-i} contains 7 vertices, the dominating set should contain at least one vertex of degree 3, and the subgraph of G_{-i} obtained after removing a vertex of degree 3 and its neighbors consists of a line of three vertices; clearly, the middle vertex of the line, which in all cases has degree 3 in G_{-i}, is the only possible choice for inclusion in the dominating set.)

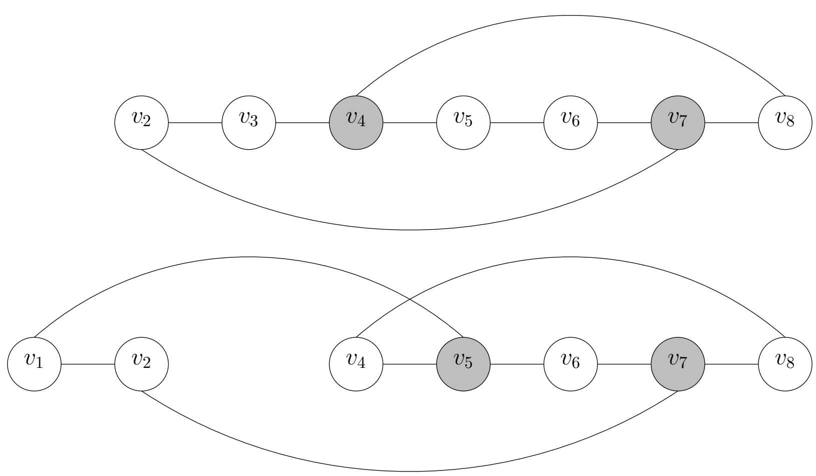

Figure 2: Graphs G_{-1} (top) and G_{-3} (bottom). Grey vertices form the only dominating sets of cardinality two.

We conclude that, in order for vertex v_i to reduce the sum of distances to 12, v_i should connect to both vertices of G that form a dominating set in G_{-i}. The claim follows by noticing that, in G, v_i is not connected to any of these two vertices, and hence cannot improve its sum of distances with a single edge swap. This concludes the proof of Theorem 5 in [1].

We have verified by exhaustive computer search that no graph with fewer than eight vertices is in sum equilibrium. Among graphs with eight vertices, Figure 1 has the fewest number 10 of edges, along with one other graph in which edge $\{v_2, v_7\}$ is replaced by $\{v_3, v_6\}$; there are also examples with 11 and 12 edges.

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References

[1] N. Alon, E. D. Demaine, M. T. Hajiaghayi, and T. Leighton. Basic network creation games. *SIAM Journal on Discrete Mathematics*, 27(2):656–668, 2013.