The number of edge colorings with no monochromatic cliques

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Abstract

Let $F(n,r,k)$ denote the maximum possible number of distinct edge-colorings of a simple graph on n vertices with r colors, which contain no monochromatic copy of K_k. It is shown that for every fixed k and all $n > n_0(k)$, $F(n,2,k) = 2^{t_{k-1}(n)}$ and $F(n,3,k) = 3^{t_{k-1}(n)}$, where $t_{k-1}(n)$ is the maximum possible number of edges of a graph on n vertices with no K_k, (determined by Turán's Theorem). The case $r = 2$ settles a conjecture of Yuster. On the other hand, for every fixed $r > 3$ and $k > 2$, the function $F(n,r,k)$ is exponentially bigger than $r^{t_{k-1}(n)}$. The proofs are based on Szemerédi's regularity lemma together with some additional tools in Extremal Graph Theory, and provide an example of a precise result proved by applying this lemma.

1 Introduction

Given a graph G, denote by $F(G,r,k)$ the number of distinct edge colorings of G with r colors which contain no monochromatic copy of K_k, i.e., a complete graph on k vertices. Let

$$F(n,r,k) = \max\big\{F(G,r,k) \mid G \text{ is a graph on } n \text{ vertices}\big\}.$$

In this paper we are interested in the behavior of $F(n,r,k)$ for fixed r and $k > 2$ and sufficiently large n. Denote by $T_{k-1}(n)$ the complete $(k-1)$-partite graph on n vertices with class sizes as equal as possible, usually called the *Turán graph* (with parameters n and $k-1$). Let $t_{k-1}(n)$ be the number of edges in $T_{k-1}(n)$. Then Turán's theorem tells us that if G is a K_k-free graph of order n then the number of edges of G, $e(G)$, satisfies $e(G) \le t_{k-1}(n)$, with equality iff $G = T_{k-1}(n)$. It is trivial to see that $F(n,r,k) \ge r^{t_{k-1}(n)}$, since every r-edge coloring of the corresponding Turán graph contains

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no monochromatic k-clique. Therefore, it is natural to ask if this lower bound reflects the correct behavior of $F(n,r,k)$. Indeed, Erdős and Rothschild [4] (see also [5]) conjectured over twenty years ago that $F(n,2,3)=2^{\lfloor n^2/4\rfloor}$ for all large enough n. This conjecture was proved by Yuster [9]. He also conjectured in [9] that the equality $F(n,2,k)=2^{t_{k-1}(n)}$ holds for all values of $k>3$, provided n is sufficiently large. In this paper we obtain the following result, which in particular proves Yuster's conjecture.

Theorem 1.1 *Let $k\geq 2$ be an integer and let $r=2$ or $r=3$. Then there exists $n(k)$, such that every graph G of order $n>n(k)$ has at most $r^{t_k(n)}$ edge colorings with r colors that have no monochromatic copy of K_{k+1}. Moreover, the only graph on n vertices for which $F(G,r,k+1)=r^{t_k(n)}$ is the Turán graph $T_k(n)$.*

In this paper we present the proof of this theorem only for $r=3$, which is the more difficult case. It is rather straightforward to make the necessary changes in this proof to obtain the result for $r=2$ and we will omit it here.

This result does not extend to more than three colors, and indeed for $r>3, k>1$ and all sufficiently large n, there is a graph G on n vertices for which $F(G,r,k+1)\gg r^{t_k(n)}$. We will prove the following results.

Theorem 1.2 $F(n,4,3)=\left(3^{1/2}2^{1/4}\right)^{\binom{n}{2}+o(n^2)}$, $F(n,4,4)=\left(3^{8/9}\right)^{\binom{n}{2}+o(n^2)}$.

Theorem 1.3 *For every fixed $r\geq 4$ and $k>1$, the function $F(n,r,k+1)$ satisfies the following*

$$\textit{If}\ \ \frac{r(k-1)}{k}>e\ \ \textit{then}\ F(n,r,k+1)\leq\left(r\frac{k-1}{k}\right)^{\frac{n^2}{2}+o(n^2)}. \tag{1}$$

$$\textit{If}\ r\geq k\ \textit{then}\ \ F(n,r,k+1)\geq\left(r\frac{k-1}{k}-2\sqrt{r\log r}\right)^{(1-\frac{1}{r})(\frac{n^2}{2}+o(n^2))}. \tag{2}$$

$$\textit{For}\ \ n\gg k+r\gg 1\qquad F(n,r,k+1)=\left(r\frac{k-1}{k}(1+o(1))\right)^{\frac{n^2}{2}} \tag{3}$$

where the $o(1)$ tends to 0 as max $\{k,r\}$ tends to infinity.

The proof of Theorem 1.1 is presented in the next two sections. It uses several tools from Extremal Graph Theory, including the regularity lemma of Szemerédi, and provides one of the rare examples in which this lemma is used to prove a precise result (for all large n). The proofs of Theorems 1.2 and 1.3 are given in Section 4, and the final Section 5 contains some concluding remarks.

2 The structure of graphs with many 3-edge colorings

As we already mentioned, we will only give the proof of Theorem 1.1 for $r = 3$, as the case $r = 2$ can be treated similarly. As the first step in the proof, we determine here the structure of any potential counterexamples. Our aim is to show that every such counterexample must be almost k-partite. For integers k and t let $K_{k+1}(t)$ be the complete $(k+1)$-partite graph with t vertices in every class. We obtain the following slightly more general result.

Lemma 2.1 *Let k and t be two positive integers. Then, for all $\delta > 0$ there exists n_0, such that if G is a graph of order $n > n_0$ which has at least $3^{t_k(n)}$ $K_{k+1}(t)$-free 3-edge colorings then there is a partition of the vertex set $V(G) = V_1 \cup \cdots \cup V_k$ such that $\sum_i e(V_i) < \delta n^2$.*

To prove this lemma we use an approach similar to the one from [2], which is based on two important tools, the Simonovits stability theorem and the Szemerédi Regularity Lemma. The stability theorem ([7], see also [3], p. 340) asserts that a K_{k+1}-free graph with almost as many edges as the Turán graph is essentially k-partite. The precise statement follows.

Theorem 2.2 *For every $\alpha > 0$ there exists $\beta > 0$, such that any K_{k+1}-free graph on m vertices with at least $\left(1 - \frac{1}{k}\right)m^2/2 - \beta m^2$ edges has a partition of the vertex set $V = V_1 \cup \cdots \cup V_k$ with $\sum_i e(V_i) < \alpha m^2$.*

Our second tool is a multicolored version of Szemerédi's Regularity Lemma. Here we will just give the definitions and the statement of the result that we require. For more details, we refer the interested reader to the excellent survey of Komlós and Simonovits [6], which discusses various applications of this powerful result.

Let $G = (V, E)$ be a graph, and let A and B be two disjoint subsets of $V(G)$. If A and B are non-empty, define the *density of edges* between A and B by

$$d(A, B) = \frac{e(A, B)}{|A||B|}.$$

For $\epsilon > 0$ the pair (A, B) is called *ϵ-regular* if for every $X \subset A$ and $Y \subset B$ satisfying $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$ we have

$$|d(X, Y) - d(A, B)| < \epsilon.$$

Intuitively, such a pair (A, B) behaves approximately as if each possible edge between A and B had been chosen randomly with probability $d(A, B)$.

An *equitable partition* of a set V is a partition of V into pairwise disjoint classes $V_1, \cdots, V_m$ of almost equal size, i.e., $\big||V_i| - |V_j|\big| \le 1$ for all i, j. An equitable partition of the set of vertices V of G into the classes $V_1, \cdots, V_m$ is called *ϵ-regular* if $|V_i| \le \epsilon|V|$ for every i and all but at most $\epsilon\binom{m}{2}$ of the pairs (V_i, V_j) are ϵ-regular.

A rough statement of the Regularity Lemma says that any graph can be approximated by a multipartite graph with a bounded number of classes, where the distribution of the edges between classes is in some sense as in a random graph. More precisely, Szemerédi [8] proved the following.

Lemma 2.3 *For every $\epsilon > 0$, there is an integer $M(\epsilon) > 0$ such that for every graph G of order $n > M$ there is a ϵ-regular partition of the vertex set of G into m classes, for some $1/\epsilon \leq m \leq M$.*

To prove Lemma 2.1 we will need a colored version of the Regularity Lemma. Its proof is a straightforward modification of the proof of the original result (see, e.g., [6] for details).

Lemma 2.4 *For every $\epsilon > 0$ and integer r, there exists an $M(\epsilon, r)$ such that if the edges of a graph G of order $n > M$ are r-colored $E(G) = E_1 \cup \cdots \cup E_r$, then there is a partition of the vertex set $V(G) = V_1 \cup \cdots \cup V_m$, with $m \leq M$, which is ϵ-regular simultaneously with respect to all graphs $G_i = (V, E_i)$ for $1 \leq i \leq r$.*

A useful notion associated with a regular partition is that of a *cluster graph.* Suppose that G is a graph with an ϵ-regular partition $V = V_1 \cup \cdots \cup V_m$, and $\eta > 0$ is some fixed constant (to be thought of as small, but much larger than ϵ.) The cluster graph $H(\eta)$ is defined on the vertex set $\{1, \cdots, m\}$ by declaring ij to be an edge if (V_i, V_j) is an ϵ-regular pair with edge density at least η. From the definition, one might expect that if a cluster graph contains a copy of a fixed clique then so does the original graph. This is indeed the case, as established in the following well-known lemma (see [6]), which says more generally that if the cluster graph contains a K_{k+1} then, for any fixed t, the original graph contains a complete $(k+1)$-partite graph $K_{k+1}(t)$.

Lemma 2.5 *For every $\eta > 0$ and integers $k, t > 0$ there exist an $0 < \epsilon = \epsilon(\eta, k, t)$ and $n_0 = n_0(\eta, k, t)$ with the following property. Suppose that G is a graph of order $n > n_0$ with an ϵ-regular partition $V = V_1 \cup \cdots \cup V_m$. Let $H(\eta)$ be the cluster graph of the partition. If $H(\eta)$ contains a K_{k+1} then G contains a $K_{k+1}(t)$.*

Having finished all the preliminaries, we are now ready to prove the lemma, which tells us the structure of any potential counterexample to Theorem 1.1.

Proof of Lemma 2.1. Suppose that a graph $G = (V, E)$ has n vertices and at least $3^{t_k(n)}$ $K_{k+1}(t)$-free 3-edge colorings. Fix some $\eta > 0$ (which we will later choose to be appropriately small) and let ϵ be such as to satisfy the assertion of Lemma 2.5. We may also choose $\epsilon < \eta$.

Consider any fixed 3-edge coloring of G without a monochromatic $K_{k+1}(t)$. By applying Lemma 2.4 we get a partition $V = V_1 \cup \cdots \cup V_m$ with respect to which the graph of each of the three colors is ϵ-regular. Let H_1, H_2, and H_3 be the corresponding cluster graphs on the vertex set $\{1, \cdots, m\}$. To simplify the notation we suppress the dependence on η here and in the rest of the proof. By Lemma 2.5 each cluster graph is K_{k+1}-free and thus by Turán's theorem it has at most $t_k(m)$ edges.

First we bound the number of 3-edge colorings of G that could give rise to this particular partition and these cluster graphs. Note that by definition, there are at most $4\epsilon\binom{n}{2}$ edges that either lie within some class of the partition or join a pair of classes that is not regular with respect to some color. Also there are at most $3\eta\binom{n}{2}$ edges that join a pair of classes in which their color has density smaller than η. Altogether, this gives no more than $7\eta\binom{n}{2} < 4\eta n^2$ edges. There are at most $\binom{n^2/2}{4\eta n^2}$ ways to choose this set of edges and they can be colored in at most $3^{4\eta n^2}$ different ways. Now, for any pair $1 \le i \ne j \le m$ consider the remaining edges between V_i and V_j. If ij is an edge in exactly s of the cluster graphs, where $0 \le s \le 3$, then every remaining edge between V_i and V_j has only s possible colors. Clearly $e(V_i, V_j) \le (n/m)^2$, so there are at most $s^{(n/m)^2}$ ways of coloring these edges. Let e_s denote the number of pairs (i,j), $i < j$ that are edges in exactly s of the cluster graphs and let $p_s = 2e_s/m^2$. Then, by the above discussion, the number of potential 3-edge colorings of G that could give this vertex partition and these cluster graphs is at most

$$\binom{n^2/2}{4\eta n^2} 3^{4\eta n^2}\Big(1^{e_1}2^{e_2}3^{e_3}\Big)^{n^2/m^2} \le 2^{H(8\eta)n^2/2}3^{4\eta n^2}\Big(2^{p_2}3^{p_3}\Big)^{n^2/2} < 3^{\big(H(8\eta)+8\eta\big)n^2/2}\Big(2^{p_2}3^{p_3}\Big)^{n^2/2}.$$

Here we use the well known estimate $\binom{a}{xa} \le 2^{H(x)a}$ for $0 < x < 1$, where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the entropy function. As we already mentioned, by Turán's theorem $e(H_i) \le t_k(m)$ for all i. Thus

$$p_1 + 2p_2 + 3p_3 = \frac{e_1 + 2e_2 + 3e_3}{m^2/2} = \frac{e(H_1) + e(H_2) + e(H_3)}{m^2/2} \le 3\frac{k-1}{k}.$$

From this we deduce that $p_2 \le \frac{3}{2}\big(\frac{k-1}{k} - p_3\big)$. Since $2 < 3^{7/11}$ this implies that

$$2^{p_2}3^{p_3} \le 3^{7p_2/11+p_3} \le 3^{\big(21\frac{k-1}{k}+p_3\big)/22}.$$

Next we claim that there must be some choice of our initial coloring for which $p_3 \ge \frac{k-1}{k} - 200\eta - 22H(8\eta)$. Indeed, suppose that $p_3 < \frac{k-1}{k} - 200\eta - 22H(8\eta)$ for all $K_{k+1}(t)$-free 3-edge colorings of G. Then, by the above inequality we have $2^{p_2}3^{p_3} < 3^{\frac{k-1}{k}-9\eta-H(8\eta)}$. Note that M is a constant and there are at most n^{M+1} partitions of the vertex set of G into at most M parts. Also, for every such partition there are at most $2^{3M^2/2}$ choices for cluster graphs H_1, H_2 and H_3. All this implies that, for sufficiently large n, the total number of possible $K_{k+1}(t)$-free 3-edge colorings is bounded by

$$n^{M+1}2^{3M^2/2}3^{\big(H(8\eta)+8\eta\big)n^2/2}\Big(2^{p_2}3^{p_3}\Big)^{n^2/2} < n^{M+1}2^{3M^2/2}3^{\big(H(8\eta)+8\eta\big)n^2/2}\Big(3^{\frac{k-1}{k}-9\eta-H(8\eta)}\Big)^{n^2/2} < 3^{t_k(n)},$$

which is a contradiction.

So we may suppose that $p_3 \ge \frac{k-1}{k} - 200\eta - 22H(8\eta)$ for some choice of initial coloring. Fix the partition $V_1 \cup \cdots \cup V_m$ together with the cluster graphs H_i which correspond to this particular coloring. Then we have

$$e_1 + e_2 = \big(p_1 + p_2\big)m^2/2 \le \big(p_1 + 2p_2\big)m^2/2 \le \Big(3\frac{k-1}{k} - 3p_3\Big)m^2/2 \le 300\eta m^2 + 33H(8\eta)m^2.$$

Let H be the graph of edges that are in all three cluster graphs. By definition, H is a K_{k+1}-free graph with $e_3 = p_3 m^2/2 \ge \big(1 - \frac{1}{k}\big)m^2/2 - (100\eta + 11H(8\eta))m^2$ edges on the vertex set $\{1, \cdots, m\}$. Suppose that $\delta > 0$ is given. Since $H(8\eta)$ tends to zero together with η, by Theorem 2.2 we could have chosen η small enough so that there is a partition $U_1 \cup \cdots \cup U_k$ of the set $\{1, \cdots, m\}$ which satisfies $\sum_i e_H(U_i) < (\delta - 304\eta - 33H(8\eta))m^2$. Let $W_i = \cup_{j \in U_i} V_j$, for $1 \le i \le k$. Then

$$\sum_{i=1}^{k} e_G(W_i) \le 4\eta n^2 + (n/m)^2\Big(\sum_{i=1}^{k} e_H(U_i) + e_1 + e_2\Big) < \delta n^2$$

and we have found the partition that satisfies the assertion of the lemma. □

3 Proof of Theorem 1.1

In this section we complete the proof of our first theorem. We start by recalling some notation and facts. $T_k(n)$ denotes the Turán graph, which is a complete k-partite graph on n vertices with class sizes as equal as possible, and $t_k(n)$ is the number of edges in $T_k(n)$. Let $\delta_k(n)$ denote the minimum degree of $T_k(n)$. For future reference we record the following simple observations.

$$t_k(n) = t_k(n-1) + \delta_k(n), \qquad \delta_k(n) = n - \lceil n/k \rceil, \qquad \frac{k-1}{k}n^2/2 - k < t_k(n) \le \frac{k-1}{k}n^2/2.$$

We also need one additional easy lemma, before we present the proof of Theorem 1.1.

Lemma 3.1 *Let G be a graph and let $W_1, \cdots, W_k$ be subsets of vertices of G such that for every $i \neq j$ and every pair of subsets $X_i \subseteq W_i, |X_i| \ge 10^{-k}|W_i|$ and $X_j \subseteq W_j, |X_j| \ge 10^{-k}|W_j|$ there are at least $\frac{1}{10}|X_i||X_j|$ edges between X_i and X_j in G. Then G contains a copy of K_k with one vertex in each set W_i.*

Proof. We use induction on k. For $k = 1$ and $k = 2$ the statement is obviously true. Suppose it is true for $k-1$ and let $W_1, \cdots, W_k$ be the subsets of vertices of G which satisfy the conditions of the lemma.

For every $1 \le i \le k-1$ denote by W_k^i the subset of vertices in W_k which have less than $|W_i|/10$ neighbors in W_i. By definition, we have $e(W_k^i, W_i) < |W_k^i||W_i|/10$ and therefore $|W_k^i| < 10^{-k}|W_k|$. Thus we deduce that $\big|\bigcup_{i=1}^{k-1} W_k^i\big| < (k-1)10^{-k}|W_k| < |W_k|/2$. So in particular there exists a vertex v in W_k which does not belong to $\bigcup_{i=1}^{k-1} W_k^i$. For every $1 \le i \le k-1$ let W_i' be the set of neighbors of v in W_i. By definition, W_i' has size at least $|W_i|/10$. Note that for every pair of subsets $X_i \subseteq W_i'$ and $X_j \subseteq W_j'$ with sizes $|X_i| \ge 10^{-(k-1)}|W_i'| \ge 10^{-k}|W_i|$ and $|X_j| \ge 10^{-(k-1)}|W_j'| \ge 10^{-k}|W_j|$, G contains at least $\frac{1}{10}|X_i||X_j|$ edges between X_i and X_j. By the induction hypothesis there exists a copy of K_{k-1} with one vertex in each W_i', for $1 \le i \le k-1$. This copy, together with the vertex v, forms a complete graph of order k with one vertex in each W_i. □

Proof of Theorem 1.1. Let n_0 be large enough to guarantee that the assertion of Lemma 2.1 holds for $\delta = 10^{-8k}$. Suppose that G is a graph on $n > n_0^2$ vertices with at least $3^{t_k(n)+m}$ K_{k+1}-free 3-edge colorings, for some $m \ge 0$. Our argument is by induction with an improvement at every step. More precisely, we will show that if G is not the corresponding Turán graph then it contains a vertex x such that $G - x$ has at least $3^{t_k(n-1)+m+1}$ K_{k+1}-free 3-edge colorings. Iterating, we obtain a graph on n_0 vertices with at least $3^{t_k(n_0)+m+n-n_0} > 3^{n_0^2}$ 3-edge colorings. But a graph on n_0 vertices has at most $n_0^2/2$ edges and hence at most $3^{n_0^2/2}$ 3-edge colorings. This contradiction will prove the theorem for $n > n_0^2$.

Recall that $\delta_k(n)$ denotes the minimum degree of $T_k(n)$, and $t_k(n) = t_k(n-1) + \delta_k(n)$. If G contains a vertex x of degree less than $\delta_k(n)$, then the edges incident with x have at most $3^{\delta_k(n)-1}$

colorings. Thus $G-x$ should have at least $3^{t_k(n-1)+m+1}$ K_{k+1}-free 3-edge colorings and we are done. Hence we may and will assume that all the vertices of G have degree at least $\delta_k(n)$.

Consider a partition $V_1 \cup \cdots \cup V_k$ of the vertex set of G which minimizes $\sum_i e(V_i)$. By our choice of n_0 in Lemma 2.1, we have that $\sum_i e(V_i) < 10^{-8k}n^2$. Note that if $|V_i| > \big(1/k + 10^{-6k}\big)n$, for some i, then every vertex in V_i has at least $\delta_k(n) - \big(\frac{k-1}{k}n - 10^{-6k}n\big) \geq 10^{-6k}n - 1$ neighbors in V_i. Thus $\sum_i e(V_i) > (10^{-6k}n - 1)(1/k + 10^{-6k})n/2 > 10^{-8k}n^2$, a contradiction. Therefore, $|V_i| - n/k \leq 10^{-6k}n$ for every i and also $|V_i| = n - \sum_{j\neq i}|V_j| \geq n/k - (k-1)10^{-6k}n$. So for every i we have $\big||V_i| - n/k\big| < 10^{-5k}n$. Let $\mathcal{C}$ denote the set of all possible K_{k+1}-free 3-colorings of the edges of G. We will refer to the colors as red, blue and green.

First consider the case when there is some vertex with many neighbors in its own class of the partition, say $x \in V_1$ with $|N(x) \cap V_1| > n/(300k)$. Our choice of partition guarantees that in this case $|N(x) \cap V_i| > n/(300k)$ also for all $2 \leq i \leq k$, or by moving x to another part we could reduce $\sum_i e(V_i)$. Let $\mathcal{C}_1$ be the subset of all the colorings in which for every i there is a subset $W_i \subset V_i$ with $|W_i| \geq n/(10^3k)$ such that all the edges from x to $\bigcup_i W_i$ have the same color, and let $\mathcal{C}_2 = \mathcal{C} - \mathcal{C}_1$.

Consider a coloring of G belonging to $\mathcal{C}_1$. Then, by definition, we have sets $W_i \subset V_i$ with $|W_i| \geq n/(10^3k)$ for each $1 \leq i \leq k$ such that all edges from x to $\cup_i W_i$ have the same color, say red. There is no red K_{k+1}, so by Lemma 3.1 there is a pair (i,j) and subsets $X_i \subset W_i$, $X_j \subset W_j$ with $|X_i| \geq 10^{-k}|W_i|$ and $|X_j| \geq 10^{-k}|W_j|$ with at most $\frac{1}{10}|X_i||X_j|$ red edges between X_i and X_j. Since there are at most $|X_i||X_j|$ edges between these two sets, we have at most $2^{|X_i||X_j|}$ ways to color the remaining edges between X_i and X_j using blue and green colors. There are at most $\binom{k}{2}\binom{|V_i|}{|X_i|}\binom{|V_j|}{|X_j|} < 2^{2n}$ ways to choose X_i and X_j and at most $\binom{|X_i||X_j|}{|X_i||X_j|/10} \leq 2^{H(0.1)|X_i||X_j|}$ ways to choose the red edges between X_i and X_j. In addition, from the structure of G we know that there are at most $t_k(n) + 10^{-8k}n^2 - |X_i||X_j|$ other edges in this graph, so the number of colorings in $\mathcal{C}_1$ can be bounded as follows

$$\begin{aligned}
|\mathcal{C}_1| &\leq 3^{t_k(n)+10^{-8k}n^2-|X_i||X_j|}\, 2^{2n} 2^{H(0.1)|X_i||X_j|} 2^{|X_i||X_j|} \leq 3^{t_k(n)+10^{-8k}n^2-|X_i||X_j|}\, 2^{2n}\, 2^{(3/2)|X_i||X_j|} \\
&= 3^{t_k(n)+10^{-8k}n^2}\, 2^{2n}\, (\sqrt{8}/3)^{|X_i||X_j|} \leq 3^{t_k(n)+10^{-8k}n^2}\, 2^{2n}\, (\sqrt{8}/3)^{10^{-2k-6}k^{-2}n^2} \\
&< 3^{t_k(n)+10^{-8k}n^2}\, 2^{2n} \left(3^{-0.01}\right)^{10^{-2k-6}k^{-2}n^2} = 3^{t_k(n)}\, 2^{2n}\, 3^{-(10^{-2k-8}k^{-2}-10^{-8k})n^2} \\
&\ll 3^{t_k(n)-1}
\end{aligned}$$

In this estimate we used the facts that $H(1/10) < 1/2$, $|X_i|, |X_j| \geq n/(k10^{k+3})$, $\sqrt{8}/3 < 3^{-0.01}$ and that $10^{-2k-8}k^{-2} - 10^{-8k} > 0$ for all $k \geq 2$.

By the above discussion, $|\mathcal{C}_2|$ contains at least $|\mathcal{C}| - |\mathcal{C}_1| \geq 3^{t_k(n)+m-1}$ colorings of G. Now we consider one of them. By definition, there are classes V_i, V_j, and V_l, so that there are at most $n/(10^3k)$ red edges from x to V_i, at most $n/(10^3k)$ green edges from x to V_j and at most $n/(10^3k)$ blue edges from x to V_l. Recall that $|N(x) \cap V_i| > n/(300k)$ for all $1 \leq i \leq k$, so we can not have $i = j = l$. Suppose first that i, j, and l are all distinct. Since the size of V_i is at most $(1/k + 10^{-5k})n$, we obtain that there are at most $\binom{(1/k+10^{-5k})n}{n/(10^3k)}$ ways to pick the red edges between x and V_i. Since the remaining edges can only have color blue or green we obtain that the number of colorings of edges between x and V_i is bounded by

$$\binom{(1/k+10^{-5k})n}{\frac{n}{10^3k}} 2^{\left(1/k+10^{-5k}\right)n} \le 2^{\left(H(0.001)+1\right)\left(1/k+10^{-5k}\right)n} \le 2^{1.02\left(1/k+10^{-5k}\right)n},$$

since $H(0.001) < 0.02$. This estimate is valid for the number of colorings of edges between x and V_j, and between x and V_l as well. Note that in addition x is incident to at most $n - |V_i| - |V_j| - |V_l| \le (\frac{k-3}{k} + 3 \cdot 10^{-5k})n$ other edges, which can have all three colors. Using the above inequalities together with the facts that $2^{3.06} < 3^{1.95}$ and $4/(100k) > 5 \cdot 10^{-5k}$ for all $k \ge 2$, we obtain that the number of colorings of the edges incident at x is at most

$$\binom{k}{3}\left(2^{1.02\left(1/k+10^{-5k}\right)n}\right)^3 3^{\left(\frac{k-3}{k}+3\cdot10^{-5k}\right)n} < 3^{\left(\frac{2}{k}-\frac{5}{100k}+2\cdot10^{-5k}\right)n} 3^{\left(\frac{k-3}{k}+3\cdot10^{-5k}\right)n} \le 3^{\left(\frac{k-1}{k}-\frac{1}{100k}\right)n}.$$

Next suppose that $i = j \ne l$. Then again there are at most $2^{1.02(1/k+10^{-5k})n}$ colorings of the edges between x and V_l and there are at most

$$\binom{(1/k+10^{-5k})n}{\frac{n}{10^3k}}^2 \le 2^{2H(0.001)\left(1/k+10^{-5k}\right)n} \le 2^{0.04\left(1/k+10^{-5k}\right)n}$$

ways to choose the red and the green edges from x to V_i. Altogether, it gives at most $2^{1.06(1/k+10^{-5k})n}$ colorings of the edges between x and $V_i \cup V_l$. Also x is incident to at most $n - |V_i| - |V_l| \le (\frac{k-2}{k} + 2 \cdot 10^{-5k})n$ other edges which can be colored arbitrarily. Therefore, since $2^{1.06} < 3^{0.95}$, we can bound the number of colorings of the edges incident at x again by

$$\binom{k}{2} 2^{1.06\left(1/k+10^{-5k}\right)n} 3^{\left(\frac{k-2}{k}+2\cdot10^{-5k}\right)n} < 3^{\left(\frac{1}{k}-\frac{5}{100k}+10^{-5k}\right)n} 3^{\left(\frac{k-2}{k}+2\cdot10^{-5k}\right)n} < 3^{\left(\frac{k-1}{k}-\frac{1}{100k}\right)n}.$$

But we had that $|\mathcal{C}_2| \ge 3^{t_k(n)+m-1}$. Hence the number of K_{k+1}-free 3-edge colorings of $G - x$ is at least

$$3^{t_k(n)+m-1-(\frac{k-1}{k}-1/100k)n} \gg 3^{t_k(n-1)+m+1}.$$

This completes the induction step in the first case.

Now we may assume that every vertex has degree at most $n/(300k)$ in its own class. We may suppose that G is not k-partite, or else by Turán's theorem $e(G) \le t_k(n)$ and therefore $|\mathcal{C}| \le 3^{t_k(n)}$ with equality only for $G = T_k(n)$. So, without loss of generality, we suppose that G contains an edge xy with $x, y \in V_1$. Let $\mathcal{C}_1$ denote the set of all K_{k+1}-free 3-edge colorings of G in which there are sets $W_i \subset V_i, |W_i| \ge n/(10^3k)$ for every $2 \le i \le k$ such that all the edges from both x and y to $\bigcup_i W_i$ and the edge xy itself have the same color. Let $\mathcal{C}_2 = \mathcal{C} - \mathcal{C}_1$ denote the remaining colorings.

Consider a coloring of G from $\mathcal{C}_1$ and assume without loss of generality that xy is colored red. Then, by definition, we have sets $W_i \subset V_i$ with $|W_i| \ge n/(10^3k)$ for each $2 \le i \le k$ such that all edges from both x and y to $\bigcup_i W_i$ are red. There is no red K_{k+1} in this coloring and therefore there is no red K_{k-1} with one vertex in each set W_i. Thus, by Lemma 3.1, there is a pair (i, j) and subsets

$X_i \subset W_i$, $X_j \subset W_j$ with $|X_i| \geq 10^{-(k-1)}|W_i|$ and $|X_j| \geq 10^{-(k-1)}|W_j|$ with at most $\frac{1}{10}|X_i||X_j|$ red edges between X_i and X_j. Arguing exactly as before in the first case we can prove that $|\mathcal{C}_1| < 3^{t_k(n)-1}$ and thus $|\mathcal{C}_2| \geq 3^{t_k(n)+m-1}$.

Next consider a coloring of G from $\mathcal{C}_2$ and suppose again that xy is red. Then there is some class V_i, $i \leq 2 \leq k$, in which x and y have at most $n/(10^3k)$ common neighbors to which they are both joined by red edges. Note that for any other vertex z in V_i, we can not color both edges zx and zy red. Therefore we have at most 8 possibilities to color these edges. Since there are at most $(1/k + 10^{-5k})n$ vertices in V_i we have at most $8^{(1/k+10^{-5k})n}$ ways to color such edges and at most

$$\binom{(1/k + 10^{-5k})n}{\frac{n}{10^3k}} \leq 2^{H(0.001)\left(1/k+10^{-5k}\right)n} \leq 2^{0.02\left(1/k+10^{-5k}\right)n}$$

possibilities to choose a set of red common neighbors of x and y in V_i. Using that $2^{3.02} < 3^{1.96}$, we obtain that there are at most

$$2^{0.02\left(1/k+10^{-5k}\right)n}\, 8^{\left(1/k+10^{-5k}\right)n} = 2^{3.02\left(1/k+10^{-5k}\right)n} < 3^{2\left(1/k-2/(100k)+10^{-5k}\right)n}$$

ways to color edges from x,y to V_i. Note that, since the degree of x and y in V_1 is at most $n/(300k)$ we have that the number of edges from x, y to $\bigcup_{j \neq i} V_i$ is bounded by $2(\frac{k-2}{k} + 2 \cdot 10^{-5k}) + 2n/(300k)$ Even if all these edges can be colored arbitrarily, since $1/(300k) > 3 \cdot 10^{-5k}$ and we have $k-1$ choices for index i, we can bound the number of colorings of the edges incident at x and y by

$$(k-1)\, 3^{2\left(1/k-2/(100k)+10^{-5k}\right)n}\, 3^{2\left(\frac{k-2}{k}+\frac{1}{300k}+2\cdot 10^{-5k}\right)n} < 3^{2\left(\frac{k-1}{k}-1/100k\right)n}.$$

But we know that $|\mathcal{C}_2| \geq 3^{t_k(n)+m-1}$. Thus the number of K_{k+1}-free 3-edge colorings of $G - \{x, y\}$ is at least

$$3^{t_k(n)+m-1-2(\frac{k-1}{k}-1/100k)n} \gg 3^{t_k(n-2)+m+2}.$$

This completes two induction steps for the second case and proves the theorem. $\Box$

Finally, we remark that it is possible to modify the argument to apply to the general situation of finding the number of H-free colorings, where H is any edge-color-critical graph. We say that a graph H with chromatic number $\chi(H) = k+1$ is *edge-color-critical* if there is some edge e of H for which $\chi(H - e) = k$. Then the following generalization holds.

Theorem 3.2 *Let H be an edge-color-critical graph with chromatic number $k+1 \geq 3$. Let $r = 2$ or $r = 3$. Then there exists $n(H)$, such that every graph G of order $n > n(H)$ has at most $r^{t_k(n)}$ edge colorings with r colors having no monochromatic copy of H, with equality only for $G = T_k(n)$.*

Sketch of proof. Again we just give the argument for $r = 3$. It is known (see, e.g., [7]) for such H that, for sufficiently large n, $T_k(n)$ is the unique H-free graph on n vertices with as many edges as possible. Note that if H has t vertices, then it is certainly contained in $K_{k+1}(t)$, so if a coloring is H-free it is also $K_{k+1}(t)$-free. Thus for sufficiently large n we have, by Lemma 2.1, that if a graph

G on n vertices has at least $3^{t_k(n)}$ H-free 3-edge colorings then there is a partition of the vertex set $V(G) = V_1 \cup \cdots \cup V_k$ such that $\sum_i e(V_i) = o(n^2)$.

To apply the rest of our arguments, we need the following generalization of Lemma 3.1, whose proof is essentially the same as that of Lemma 2.5 (see, e.g., [6]).

Lemma 3.3 *For any $\alpha > 0$ and any integers $t, k > 0$ there exists $\beta > 0$ such that the following holds. Let G be a graph, and let $W_1, \cdots, W_k$ be subsets of vertices of G such that for every $i \neq j$ and pair of subsets $X_i \subseteq W_i, |X_i| \geq \beta|W_i|$ and $X_j \subseteq W_j, |X_j| \geq \beta|W_j|$ there are at least $\alpha|X_i||X_j|$ edges between X_i and X_j in G. Then G contains a copy of $K_k(t)$ with t vertices in each set W_i.* $\square$

The proof of Theorem 3.2 is now almost the same as for $H = K_{k+1}$. In the first case, when there is some vertex with high degree in its class, we use Lemma 3.3 instead of Lemma 3.1 and also the simple fact that H is a subgraph of the graph obtained by connecting the vertex x with all the vertices of $K_k(t)$. For the second case, to bound the number of colorings in $\mathcal{C}_1$ we need a slight modification. We note that H is contained in the graph obtained by adding an edge to $K_k(t)$. When we are given sets $W_i \subset V_i$ for each $2 \leq i \leq k$ such that all edges from both x and y to $\cup_i W_i$ are red, we let $W_1 = V_1 \backslash \{x, y\}$. Then we will apply Lemma 3.3 to the sets $W_1, \cdots, W_k$. There are no significant changes to the rest of the proof and we leave the remaining details to the interested reader. $\square$

For example, odd cycles C_{2t+1} are edge-color-critical with chromatic number 3, so we have the following corollary.

Corollary 3.4 *For any integer $t > 0$ there exists $n(t)$, such that for any graph G on $n > n(t)$ vertices, the number of C_{2t+1}-free 2-edge and 3-edge colorings of G is at most $2^{\lfloor n^2/4 \rfloor}$ and $3^{\lfloor n^2/4 \rfloor}$, respectively, with equality only for $G = T_2(n)$.*

4 Edge colorings with more than three colors

For two or three colors we were able to show in the previous sections that the number of K_{k+1}-free colorings was largest for the corresponding Turán graph with k color classes. However, for four or more colors this is no longer true. Moreover, it is not at all obvious how large the number of K_{k+1}-free r-edge colorings of a graph of order n can be and which graphs have the maximum number of such colorings. We start with two examples, which show that already for $r = 4$ and $k = 2, 3$ there are graphs of order n which have more than $4^{t_k(n)}$ K_{k+1}-free 4-edge colorings.

Example 1. Let G be the complete 4-partite graph on n vertices with parts of almost equal size. We will show that G has many more triangle-free 4-edge colorings than the Turán graph $T_2(n)$. Let V_1, V_2, V_3, V_4 be the classes of the partition and let $\{a, b, c, d\}$ be the set of colors. Consider the set of colorings in which every edge between V_i and V_j must have one of the colors belonging to the set $c(i, j)$, where $c(1, 2) = c(3, 4) = \{a, b, d\}$, $c(1, 3) = c(2, 4) = \{a, b, c\}$ and $c(1, 4) = c(2, 3) = \{c, d\}$. It is easy to check that there are no monochromatic triangles in any of these colorings. The number of such colorings is

$$\left(3^4 2^2\right)^{(n/4)^2+\Theta(1)} = \left(3^{1/2}2^{1/4}\right)^{n^2/2+\Theta(1)}.$$

On the other hand, the number of triangle-free 4-edge colorings of $T_2(n)$ is $4^{(n/2)^2+\Theta(1)} = 2^{n^2/2+\Theta(1)}$, which is exponentially smaller, since $2 < 3^{1/2}2^{1/4}$.

Example 2. Let G be the complete 9-partite graph of order n with parts of almost equal size. We will show that G has many more K_4-free 4-edge colorings than $T_3(n)$. To describe the colorings of G it is convenient to index the classes of the partition with the points of $\mathbb{F}_3^2$, the affine plane over the finite field with 3 elements, i.e. $V = \cup_{x \in \mathbb{F}_3^2} V_x$. For x, d in $\mathbb{F}_3^2$ with $d \neq 0$, the *line* through x in *direction* d consists of the three points $\{x, x+d, x+2d\}$. Note that d and $2d$ determine the same line, so there are precisely four lines through each point. Also, for a fixed $d \neq 0$ there are three different lines in direction d and they partition $\mathbb{F}_3^2$. Let $d_1, \ldots, d_4$ be representative directions of the four lines through any point. We consider the set of colorings with colors $\{1, 2, 3, 4\}$ where for x, y in $\mathbb{F}_3^2$ we allow an edge between V_x and V_y to have color i if the line joining x to y *does not* have direction d_i. In other words, the graph of color i respects the tripartition defined by the three lines in direction d_i, and is therefore contained in the Turán graph $T_3(n)$. It thus follows that all these colorings contain no monochromatic K_4. Note that there are precisely 3 colors available for each edge, so the number of such colorings is

$$\left(3^{\binom{9}{2}}\right)^{(n/9)^2+\Theta(1)} = \left(3^{8/9}\right)^{n^2/2+\Theta(1)}.$$

On the other hand, the number of K_4-free 4-colorings of $T_3(n)$ is $\left(4^3\right)^{(n/3)^2+\Theta(1)} = \left(2^{4/3}\right)^{n^2/2+\Theta(1)}$, which is exponentially smaller, as $2^{4/3} < 3^{8/9}$.

Next we show that the exponents in these two examples are best possible.

Proof of Theorem 1.2. The above examples give the required lower bounds, so it remains to obtain the upper bounds. We start with the proof of the upper bound on $F(n, 4, 3)$.

Consider a graph $G = (V, E)$ with n vertices and any fixed 4-edge coloring of G without a monochromatic triangle. Fix any $\eta > 0$ and let $\epsilon < \eta$ be such as to satisfy the assertion of Lemma 2.5 (with $t = 1$). By applying Lemma 2.4 we get a partition $V = V_1 \cup \cdots \cup V_m, m \leq M(\eta)$ with respect to which the graph of each of the four colors is ϵ-regular. Let $H_1, \cdots, H_4$ be the corresponding cluster graphs on the vertex set $\{1, \cdots, m\}$. By Lemma 2.5 each cluster graph is triangle-free and thus by Turán's theorem it has at most $t_2(m)$ edges.

First we bound the number of 4-edge colorings of G that could give rise to this particular partition and these cluster graphs. As in the proof of Lemma 2.1 there are at most $4\eta n^2$ edges that lie within some class of the partition, or join a pair of classes that is not regular with respect to some color, or join a pair of classes in which their color has density smaller than η. There are at most $\binom{n^2/2}{4\eta n^2}$ ways to choose this set of edges and they can be colored in at most $4^{4\eta n^2}$ different ways. For $0 \leq s \leq 4$, let e_s denote the number of pairs (i, j), $i < j$ that are edges in exactly s of the cluster graphs and let $p_s = 2e_s/m^2$. Then the number of potential 4-edge colorings of G that could give this vertex partition and these cluster graphs is at most

$$\binom{n^2/2}{4\eta n^2} 4^{4\eta n^2} \left(1^{e_1} 2^{e_2} 3^{e_3} 4^{e_4}\right)^{n^2/m^2} \le 2^{H(8\eta)n^2/2}\, 4^{4\eta n^2} \left(2^{p_2} 3^{p_3} 4^{p_4}\right)^{n^2/2}.$$

As we already mentioned, by Turán's theorem $e(H_i) \le t_2(m)$ for all i. Thus

$$p_1 + 2p_2 + 3p_3 + 4p_4 = \frac{e(H_1) + e(H_2) + e(H_3) + e(H_4)}{m^2/2} \le 2. \tag{4}$$

Now consider the graph H on $\{1, \cdots, m\}$ where (i, j) is an edge of H if it is an edge in exactly 3 of the cluster graphs. Then $e(H) = e_3$. Note that however one chooses 3 sets of size 3 from a 4 element set of colors, there is a common color in all three. This implies that H is a triangle-free graph, since every triangle in H corresponds to a triangle in one of the cluster graphs H_i. Therefore by Turán's theorem we have

$$p_3 \le 1/2. \tag{5}$$

Now we want to determine the maximum value of $2^{p_2+2p_4}3^{p_3}$ subject to equations (4) and (5). Clearly we should choose $p_1 = 0$. Setting $x = p_2 + 2p_4$, we want to maximize $x \log 2 + p_3 \log 3$, subject to $2x + 3p_3 \le 2$ and $p_3 \le 1/2$. Since $\frac{1}{3}\log 3 > \frac{1}{2}\log 2$ the maximum occurs at $p_3 = 1/2$, $x = 1/4$. Hence there are at most $2^{H(8\eta)n^2/2} 4^{4\eta n^2} \left(3^{1/2}2^{1/4}\right)^{n^2/2}$ triangle-free 4-edge colorings of G that give this vertex partition and these cluster graphs. Note that M is a constant and there are at most n^{M+1} partitions of the vertex set of G into at most M parts. Also, for every such partition there are at most $2^{4(M^2/2)}$ choices for cluster graphs H_i. Since we can choose η to be arbitrarily small, we obtain that for sufficiently large n

$$F(n, 4, 3) \le n^{M+1}\, 2^{2M^2}\, 2^{H(8\eta)n^2/2}\, 4^{4\eta n^2} \left(3^{1/2}2^{1/4}\right)^{n^2/2} \le \left(3^{1/2}2^{1/4}\right)^{n^2/2+o(n^2)}.$$

Now we obtain the upper bound on $F(n, 4, 4)$. Consider a graph $G = (V, E)$ with n vertices and any fixed 4-edge coloring of G without a monochromatic K_4. Fix any $\eta > 0$ and let $\epsilon < \eta$ be such as to satisfy the assertion of Lemma 2.5. By applying Lemma 2.4 we get a partition $V = V_1 \cup \cdots \cup V_m, m \le M(\eta)$ with respect to which the graph of each of the four colors is ϵ-regular. Let $H_1, \cdots, H_4$ be the corresponding cluster graphs on the vertex set $\{1, \cdots, m\}$. By Lemma 2.5 each cluster graph is K_4-free and thus by Turán's theorem it has at most $t_3(m)$ edges.

First we bound the number of 4-edge colorings of G that could give rise to this particular partition and these cluster graphs. Again there are at most $4\eta n^2$ edges that lie within some class of the partition, or join a pair of classes that is not regular with respect to some color, or join a pair of classes in which their color has density smaller than η. There are at most $\binom{n^2/2}{4\eta n^2}$ ways to choose this set of edges and they can be colored in at most $4^{4\eta n^2}$ different ways. For $0 \le s \le 4$, let e_s denote the number of pairs (i, j), $i < j$ that are edges in exactly s of the cluster graphs and let $p_s = 2e_s/m^2$.

Then the number of potential 4-edge colorings of G that could give this vertex partition and these cluster graphs is at most

$$\binom{n^2/2}{4\eta n^2} 4^{4\eta n^2} \left(1^{e_1} 2^{e_2} 3^{e_3} 4^{e_4}\right)^{n^2/m^2} \leq 2^{H(8\eta)n^2/2} \, 4^{4\eta n^2} \left(2^{p_2} 3^{p_3} 4^{p_4}\right)^{n^2/2}.$$

As we already mentioned, by Turán's theorem $e(H_i) \leq t_3(m)$ for all i. Thus

$$p_1 + 2p_2 + 3p_3 + 4p_4 = \frac{e(H_1) + e(H_2) + e(H_3) + e(H_4)}{m^2/2} \leq 4\frac{t_3(m)}{m^2/2} \leq 8/3.$$

As before, since $\frac{1}{3}\log 3 > \frac{1}{2}\log 2 = \frac{1}{4}\log 4$, the number of colorings is maximized when we choose p_3 as large as possible, i.e. $p_3 = 8/9$. This gives at most $2^{H(8\eta)n^2/2} 4^{4\eta n^2} \left(3^{8/9}\right)^{n^2/2}$ 4-edge colorings of G that give this vertex partition and these cluster graphs. Note that M is a constant and there are at most n^{M+1} partitions of the vertex set of G into at most M parts. Also, for every such partition there are at most $2^{4(M^2/2)}$ choices for cluster graphs H_i. Since we can choose η to be arbitrarily small, we obtain that for sufficiently large n

$$F(n,4,4) \leq n^{M+1}\, 2^{2M^2}\, 2^{H(8\eta)n^2/2}\, 4^{4\eta n^2} \left(3^{8/9}\right)^{n^2/2} \leq \left(3^{8/9}\right)^{n^2/2+o(n^2)}.$$

This completes the proof of the theorem. □

So far we obtained rather accurate estimates for the values of $F(n,4,3)$ and $F(n,4,4)$. The determination or estimation of $F(n,r,k+1)$ for all r and k seems to be a much harder problem. Indeed, it is not even clear what the correct exponent should be. In general, the statement of Theorem 1.3 gives some indication on the asymptotic behavior of $F(n,r,k+1)$, when $k+r$ is large.

The proof of Theorem 1.3 is similar to the proof of Theorem 1.2. We need the following simple lemma.

Lemma 4.1 *Let N be an integer, and let $s > e$ be a real number. Then, the maximum possible product of all elements of a sequence of at most N positive reals whose sum is at most sN is at most $s^{N/s}$.*

Proof. Let $m \leq N$ be the number of elements in the sequence. By the arithmetic-geometric inequality their product is maximized when they are all equal, and in this case the product is at most $f_m = (sN/m)^m$. The function $g(m) = \ln f_m = m\ln(sN) - m\ln m$ is increasing for all admissible values of m, as its derivative is $\ln(\frac{sN}{m}) - 1 \geq \ln s - 1 > 0$, and hence the maximum possible value of f_m for $m \leq N$ is obtained when $m = N$, supplying the desired result. □

Proof of Theorem 1.3. We start with the proof of (1). Consider a graph $G = (V,E)$ with n vertices and any fixed r-edge coloring of G without a monochromatic K_{k+1}. Fix an $\eta > 0$ and let $\epsilon < \eta$ satisfy the assertion of Lemma 2.5 with $t = 1$. By Lemma 2.4 there is a partition $V = V_1 \cup \cdots \cup V_m, m \leq M(\eta)$ with respect to which the graph of each of the r colors is ϵ-regular. Let $H_1, \cdots, H_r$ be the corresponding cluster graphs on the vertex set $\{1, \cdots, m\}$. By Lemma 2.5 each cluster graph H_i is K_{k+1}-free and thus by Turán's theorem it has at most $t_k(m)$ edges.

First we bound the number of r-edge colorings of G that give rise to this particular partition and these cluster graphs. As in the proof of Lemma 2.1 there are at most $r\eta n^2$ edges that lie within some class of the partition, or join a pair of classes that is not regular with respect to some color, or join a pair of classes in which their color has density smaller than η. There are at most $\binom{n^2/2}{r\eta n^2}$ ways to choose this set of edges and they can be colored in at most $r^{r\eta n^2}$ different ways. For $0 \leq p \leq r$, let e_p denote the number of pairs (i, j), $i < j$ that are edges in exactly p of the cluster graphs H_i. Clearly

$$\sum_{p=1}^{r} e_p \leq \binom{m}{2} < \frac{m^2}{2}.$$

Therefore, the number of potential r-edge colorings of G that give this vertex partition and these cluster graphs is at most

$$\binom{n^2/2}{r\eta n^2} r^{r\eta n^2} \Big(\prod_{j=1}^{r} j^{e_j}\Big)^{n^2/m^2} \leq 2^{H(2r\eta)n^2/2}\, r^{r\eta n^2} \prod_{j=1}^{r} j^{e_j n^2/m^2}.$$

As already mentioned, by Turán's theorem $e(H_i) \leq t_k(m)$ for all i. Thus

$$\sum_{j=1}^{r} j e_j \leq \frac{r(k-1)}{k}\frac{m^2}{2}. \tag{6}$$

It follows that the product $\prod_{j=1}^{r} j^{e_j n^2/m^2}$ is a product of $\sum_{j=1}^{r} e_j n^2/m^2 \leq \frac{n^2}{2}$ positive integers whose sum is at most

$$\frac{r(k-1)}{k}\frac{m^2}{2}\frac{n^2}{m^2} = \frac{r(k-1)}{k}\frac{n^2}{2},$$

where here we used (6). By Lemma 4.1 with $N = \frac{n^2}{2}$ and $s = \frac{r(k-1)}{k}(\ > e)$ we conclude that this product is at most $(\frac{r(k-1)}{k})^{n^2/2}$. Thus, there are at most $2^{H(2r\eta)n^2/2} r^{r\eta n^2} \Big(\frac{r(k-1)}{k}\Big)^{n^2/2}$ r-edge colorings of G with no monochromatic K_{k+1} that give this vertex partition and these cluster graphs. Recall that M is a constant, and there are at most n^{M+1} partitions of the vertex set of G into at most M parts. Also, for every such partition there are at most $2^{r(M^2/2)}$ choices for the cluster graphs H_i. Therefore,

$$F(n, r, k+1) \leq n^{M+1}\, 2^{rM^2/2}\, 2^{H(2r\eta)n^2/2}\, r^{r\eta n^2} \Big(\frac{r(k-1)}{k}\Big)^{n^2/2} \leq \Big(\frac{r(k-1)}{k}\Big)^{n^2/2 + O(\eta \log(1/\eta))n^2}.$$

Since we can choose η to be arbitrarily small, it follows that

$$F(n, r, k+1) \leq \Big(\frac{r(k-1)}{k}\Big)^{n^2/2 + o(n^2)}$$

completing the proof of (1). We note that when $\frac{r(k-1)}{k}$ is not an integer, the upper bound can be slightly improved, as the assertion of Lemma 4.1 can be improved if all the elements of the given sequence are integers.

We next prove (2). Let $G = (V, E)$ be the Turán graph $T_r(n)$, and let $V_1, V_2, \ldots, V_r$ be its color classes. Our objective is to show that G has many r-edge colorings with no monochromatic K_{k+1}. For each p, $1 \leq p \leq r$, let H_p be a copy of the Turán graph $T_k(r)$ on the set of r vertices $R = \{1, 2, \ldots, r\}$, placed randomly on R. For each fixed pair i, j of distinct members of R, let $S_{ij} = \{p : \ ij \in E(H_p)\}$ denote the set of all graphs H_p containing the edge ij. The cardinality of this set is a Binomial random variable with parameters r and $t_k(r)/\binom{r}{2} \geq \frac{k-1}{k}$. By the standard estimates for Binomial distributions (c.f., e.g., [1]. Theorem A.1.13) it follows that for each fixed $i, j \in R$, the probability that $|S_{ij}| < K$, where $K = \frac{r(k-1)}{k} - 2\sqrt{r \ln r}$, is at most $1/r^2$. Hence, with positive probability all sets $S_{i,j}$ are of cardinality at least K. The result now follows by considering all colorings of G in which every edge connecting V_i and V_j is colored by a color from S_{ij}. This establishes (2).

Finally, note that the assertion of (3) for $k \leq r$ and r large follows from (1) and (2). For $k \geq r$ and k large it follows from (1) (or the trivial fact that $F(n, r, k+1) \leq r^{\binom{n}{2}}$), and the simple lower bound $F(n, r, k+1) \geq r^{\frac{k-1}{k}\binom{n}{2}}$. $\square$

5 Concluding remarks

- Using some of the arguments in the proof of Theorem 1.3, one can prove the following.

Proposition 5.1 *For every fixed r and k, the limit*

$$\lim_{n \mapsto \infty} \big(F(n, r, k+1)\big)^{2/n^2}$$

exists, and is a positive real.

Sketch of proof. Define

$$f = \limsup_{n \mapsto \infty} \big(F(n, r, k+1)\big)^{2/n^2}.$$

Fix a small $\epsilon > 0$, and let m be a large integer satisfying $\big(F(m, r, k+1)\big)^{2/m^2} \geq f - \epsilon$. Let $G = (V, E)$ be a graph on m vertices satisfying $F(G, r, k+1) = F(m, r, k+1)$. By repeating the arguments in the proof of the last theorem we conclude that if m is sufficiently large, then at least $(f - 2\epsilon)^{m^2/2}$ distinct r-edge colorings of G with no monochromatic K_{k+1} arise from the same regular partition $V_1, V_2, \ldots, V_M$ of V and the same cluster graphs H_p on $\{1, 2, \ldots, M\}$ defined as in the last proof. Let G_1 be the graph obtained from G by deleting all edges inside the classes of the partition, all edges in irregular pairs, and all edges between sparse regular pairs. For each pair of classes i, j of G_1, define $S_{ij} = \{p : \ ij \in E(H_p)\}$. If m is sufficiently large, then there is a set $\mathcal{C}$ of at least $(f - 3\epsilon)^{m^2/2}$ distinct r-edge colorings of G_1 obtained by assigning, in all possible ways, a color from S_{ij} to each edge of G_1 between V_i and V_j. All these colorings do not contain a monochromatic K_{k+1}, since no cluster graph contains a K_{k+1}.

Suppose, now, that $n > m$, and let G' be the graph obtained from G_1 by replacing each vertex of G_1 by either $\lfloor n/m \rfloor$ or $\lceil n/m \rceil$ vertices, so that the total number of vertices of G' is n. In this way, each class V_i is replaced by a class V_i' of size at least $\lfloor n/m \rfloor |V_i|$. Every r-edge coloring of G' in which every edge between V_i' and V_j' gets a color from S_{ij} contains no monochromatic K_{k+1}. The number of these colorings is clearly at least

$$|\mathcal{C}|^{(\lfloor n/m \rfloor)^2} \geq \big(f - 3\epsilon\big)^{\frac{m^2}{2}(\lfloor n/m \rfloor)^2}.$$

This implies that for every $n > m$,

$$(F(n,r,k+1))^{2/n^2} \geq \big(f - 3\epsilon\big)^{\frac{m^2}{n^2}(\lfloor n/m \rfloor)^2}.$$

Since ϵ can be arbitrarily small it follows, by the definition of f, that

$$\lim_{n \mapsto \infty} \big(F(n,r,k+1)\big)^{2/n^2} = f,$$

completing the proof. ☐

- For every $r \geq 2$ and $k > 1$, define $f(r,k+1) = lim_{n \mapsto \infty}(F(n,r,k+1))^{2/n^2}$. This limit exists, by proposition 5.1, and trivially it is at least $r^{(k-1)/k}$ and at most r. By Theorem 1.1, $f(2,k+1) = 2^{(k-1)/k}$ for all k and $f(3,k+1) = 3^{(k-1)/k}$ for all k. By Theorem 1.2, $f(4,3) = 3^{1/2}2^{1/4}$ ($> 4^{1/2}$) and $f(4,4) = 3^{8/9}$ ($> 4^{2/3}$), and by Theorem 1.3 for large $k+r$, $f(r,k+1) = \frac{r(k-1)}{k}(1+o(1))$ with the $o(1)$ term tending to zero as $k+r$ tends to infinity.

  It is not difficult to prove that in fact for every $r \geq 4$ and every $k > 1$, $f(r,k+1)$ is strictly larger than $r^{(k-1)/k}$. To do so, one first shows, using some simple constructions following the ones described in the proof of Theorems 1.2 and 1.3, that for every $r \geq 4$, $f(r,3) > r^{1/2}$. Knowing this, we can start with the Turán graph $G = T_k(n)$ as a graph that has many r-colorings with no monochromatic K_{k+1}, and get an exponentially better example by replacing the induced subgraph of G on three of the color classes whose total number of vertices is, say, n', by the best example we have for providing a lower bound for $F(n',r,3)$. (In fact, for $k \geq 3s$ we can perform such a replacement for s pairwise disjoint triples of color classes).

- The problem of determining $f(r,k)$ for all r and k seems interesting. It may also be interesting to find a proof of Theorem 1.1 without applying the regularity lemma, in order to conclude that the assertion of the theorem holds already for values of n which are not so huge as a function of r and k. It is easy to see that the assertion fails for values of n which are smaller than, say, $r^{(k-1)/2}$, as in this case, a random r-coloring of K_n contains no monochromatic K_{k+1} with probability that exceeds 1/2, showing that for such relatively small (and yet exponential) values of n, $F(n,r,k+1) > \frac{1}{2}r^{\binom{n}{2}}$.

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