The Turán number of sparse spanning graphs

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Abstract

For a graph H, the *extremal number $ex(n, H)$* is the maximum number of edges in a graph of order n not containing a subgraph isomorphic to H. Let $\delta(H) > 0$ and $\Delta(H)$ denote the minimum degree and maximum degree of H, respectively. We prove that for all n sufficiently large, if H is any graph of order n with $\Delta(H) \leq \sqrt{n}/200$, then $ex(n, H) = \binom{n-1}{2} + \delta(H) - 1$. The condition on the maximum degree is tight up to a constant factor. This generalizes a classical result of Ore for the case $H = C_n$, and resolves, in a strong form, a conjecture of Glebov, Person, and Weps for the case of graphs. A counter-example to their more general conjecture concerning the extremal number of bounded degree spanning hypergraphs is also given.

1   Introduction

Ore [8] proved that a non-Hamiltonian graph of order n has at most $\binom{n-1}{2} + 1$ edges. Ore's theorem can be expressed as a spanning Turán-type result. For a graph H, the *extremal number $ex(n, H)$* is the maximum number of edges in a graph of order n not containing a subgraph isomorphic to H. Hence, Ore's theorem is that $ex(n, C_n) = \binom{n-1}{2} + 1$, where C_n is the cycle of order n.

Recently, Ore's theorem has been generalized to the setting of Hamilton cycles in hypergraphs. For integers $n > k > \ell \geq 0$ where $(k - \ell)|n$, the (k, ℓ)-tight cycle of order n, denoted by $C_n^{(k,l)}$, is the k-uniform hypergraph on vertex set $[n]$ and edges $\{i(k - \ell) + 1, i(k - \ell) + 2, \ldots, i(k - \ell) + k\}$ for $0 \leq i < \frac{n}{k-\ell}$ (addition modulo n). In particular, $C_n = C_n^{(2,1)}$. Extending and generalizing earlier results of Katona and Kierstead [6] and of Tuza [10], the extremal number $ex(n, C_n^{(k,l)})$ was determined by Glebov, Person and Weps in [5] for all k and ℓ, assuming n is sufficiently large and $(k - \ell)|n$. Their result extends Ore's theorem to the hypergraph setting. The extremal number $ex(n, C_n^{(k,l)})$ is of the form $\binom{n-1}{k} + ex(n - 1, P)$ where P is the $(k - 1, \ell - 1)$-tight path (defined in the obvious way) with $\lfloor k/(k - \ell) \rfloor (k - \ell) + \ell - 1$ vertices.

It is natural to try to extend Ore's result to spanning structures other than just Hamilton cycles, in both the graph and hypergraph settings. Suppose that H is a k-uniform hypergraph of order n and with, say, bounded maximum degree. It is natural to suspect that $ex(n, H) \leq$

$\binom{n-1}{k} + ex(n-1, S)$ where S is some set of $(k-1)$-uniform hypergraphs that depend on the neighborhood structure of H. A conjecture raised in [5] asserts that it suffices to take S to be the set of *links* of H. The link of a vertex v in a k-uniform hypergraph $H = (V, E)$ is the $(k-1)$-uniform hypergraph $H(v) = (V \setminus \{v\}, E_v)$ with $\{x_1, \ldots, x_{k-1}\} \in E_v$ iff $\{v, x_1, \ldots, x_{k-1}\} \in E$. For example, the links of a vertex of $C_n^{(3,1)}$ are either the graph with a single edge or the graph with two independent edges. The link of a vertex of $C_n^{(3,2)}$ is the graph P_4 (the path on four vertices). Conjecture 9 in [5] states that $ex(n, H) \leq \binom{n-1}{k} + ex(n-1, \mathcal{L})$ where $\mathcal{L}$ is the set of links of H and $ex(n-1, \mathcal{L})$ denotes the maximum number of edges in a $(k-1)$-uniform hypergraph on n vertices that contains none of the links of H. Observe that this conjecture holds for both Ore's Theorem and its aforementioned generalization to Hamilton cycles in hypergraphs. In fact, it holds with equality in these cases.

In the graph-theoretic case, the link of a vertex is just a set of singletons whose cardinality is the degree of the vertex. In this case, the aforementioned conjecture states that if H is a graph of order n with minimum degree $\delta > 0$, and bounded maximum degree, then $ex(n, H) = \binom{n-1}{2} + \delta - 1$ assuming n is sufficiently large. (Clearly, we cannot expect to have a sharp inequality, as an $(n-1)$-clique together with an additional vertex that is connected only to $\delta - 1$ vertices of the clique does not contain H as a spanning subgraph.) Indeed, our main result is a proof of this conjecture in a strong sense. We do not require the maximum degree of H to be bounded independently of n.

Theorem 1.1. *For all n sufficiently large, if H is any graph of order n with no isolated vertices and $\Delta(H) \leq \sqrt{n}/200$, then $ex(n, H) = \binom{n-1}{2} + \delta(H) - 1$.*

In our proof we make no attempt to minimize the value of n starting from which the theorem holds, although it can be worked out from the proof to be less than 100000. Also, the constant 200 in the bound for the maximum degree is not optimal and can be somewhat improved. However, more importantly, the following construction shows that it *cannot* be improved to less than $\sqrt{2}$, and hence the $O(\sqrt{n})$ bound on the maximum degree is optimal up to a constant factor. Consider the graph H with $n = k(k+6)/2+1$ vertices, consisting of k vertex-disjoint cliques of size $(n-1)/k$ each, and an additional vertex connected to some $\delta \leq (n-1)/k - 1$ vertices of the cliques. Clearly, $\Delta(H) = (n-1)/k$ and $\delta(H) = \delta$. Observe, however, that H does not have an independent set of size $k+2$. Hence, if G is the n-vertex graph obtained from K_n by removing a K_{k+2}, then H is not a spanning subgraph of G. However, G has $\binom{n}{2} - \binom{k+2}{2}$ edges, which is easily checked to be more than $\binom{n-1}{2} + \delta(H) - 1$.

Our next result shows that the conjecture of Glebov et al. is false already for 3-uniform hypergraphs. To simplify the presentation we describe one example, the same proof can provide many others.

Proposition 1.2. *Let s be a large integer, define $n = 1 + 5s$ and let $V = V_1 \cup V_2 \cup \cdots \cup V_s \cup \{x\}$ be a set of n vertices, where each V_i is a set of 5 vertices, the sets V_i are pairwise disjoint, and x is an additional vertex. Let H be the 3 uniform hypergraph on the set of vertices V, where each V_i forms a complete 3-uniform hypergraph on 5 vertices, and x is contained in a unique edge $\{x, u, v\}$*

with $u, v \in V_1$. Let $\mathcal{L} = \{H(v) : v \in V\}$ be the set of all links of H. Then $ex(n-1, \mathcal{L}) = 0$ but

$$ex(n, H) \geq \binom{n-2}{3} + \frac{4}{3}\binom{n-2}{2} = \binom{n-1}{3} + \frac{1}{3}\binom{n-2}{2} > \binom{n-1}{3} + ex(n-1, \mathcal{L}).$$

The reason that H forms a counter-example is that the link of every vertex of H besides one is a complete graph on 4 vertices, and hence any hypergraph containing two vertices whose links are 3-colorable cannot contain a spanning copy of H. Since a 3-colorable graph on a set of vertices can contain more than half of all potential edges on this set, $ex(n, H)$ is larger than the number of edges of a complete 3-uniform hypergraph on $n-1$ vertices.

The rest of this short paper is organized as follows. Theorem 1.1 is proved in Section 2, Section 3 contains the proof of Proposition 1.2, and Section 4 contains some concluding remarks. Throughout the paper we use the standard graph-theoretic terminology and notations following [2].

2 Proof of the main result

We say that two graphs G and H of the same order *pack*, if H is a spanning subgraph of the complement of G. Let $H = (W, F)$ be a given graph with n vertices and with $\Delta(H) \leq \sqrt{n}/200$. Let $G = (V, E)$ be any graph with n vertices and $n - \delta - 1$ edges, where $\delta = \delta(H)$. It suffices to prove that G and H pack. Equivalently, we construct a bijection $f : V \to W$ such that for all $(u, v) \in E$, $(f(u), f(v)) \notin F$. Throughout the proof we assume that n is larger than some absolute constant.

Before describing f, we require some notation. Let $d(v)$ denote the degree of a vertex v in G, let $N(v)$ denote the set of neighbors of v and, for a subset of vertices W, let $N[W] = W \cup (\cup_{w \in W} N(w))$. Let $V = \{v_1, \ldots, v_n\}$ where $d(v_i) \geq d(v_{i+1})$ for $i = 1, \ldots, n-1$. Let $S_1 \subset V \setminus N[v_1]$ be an independent set of vertices each with degree smaller than $2\sqrt{n}$ and with maximum possible cardinality under this restriction. In the next lemma we show that $|S_1| \geq \delta$ and thus let $B_1 \subset S_1$ denote an arbitrary subset with $|B_1| = \delta$. For $i = 2, \ldots, n$, let $S_i \subset V \setminus (N[v_i] \cup N[B_1])$ be an independent set of vertices but with the additional requirement that each $u \in S_i$ has $d(u) \leq 10$. Furthermore, we require that S_i has maximum possible cardinality under these restrictions.

Lemma 2.1. *The following bounds hold:*

1. *$d(v_1) \leq n - \delta - 1$, $d(v_2) \leq n/2$, and $d(v_i) < 2n/i$.*

2. *$|S_1| \geq \delta$ and $|S_i| \geq n/18$ for $i = 2, \ldots, n$.*

Proof. Trivially, $d(v_1) = \Delta(G) \leq |E| = n - \delta - 1$. Since $d(v_1) + d(v_2) \leq |E| + 1$, we have that $d(v_2) \leq (|E| + 1)/2 \leq n/2$. Since $\sum_{i=1}^{n} d(v_i) = 2n - 2\delta - 2 < 2n$, we have that $d(v_i) < 2n/i$.

The subgraph of G consisting of the non-neighbors of v_1 has $n - d(v_1) - 1$ vertices and at most $n - \delta - 1 - d(v_1)$ edges and therefore has an independent set of size at least $(n - d(v_1) - 1)/3$. If, say, $n - d(v_1) - 1 \geq 6\delta$, at least $(n - d(v_1) - 1)/6 \geq \delta$ of its vertices have degree at most $6 < 2\sqrt{n}$. Thus, in this case $|S_1| \geq \delta$. Otherwise, any non-neighbor of v_1 has degree at most $6\delta < 2\sqrt{n}$, and

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as the subgraph induced on them has $n - d(v_1) - 1$ vertices and at most $n - \delta - 1 - d(v_1)$ edges it has at least δ components. Selecting one vertex from each component shows that $|S_1| \geq \delta$ in this case as well.

By a similar reasoning, the subgraph of G consisting of the non-neighbors of v_i does not have more edges than vertices, and hence has an independent set of size at least $1/3$ of its cardinality. As for $i \geq 2$, its cardinality is at least $n - d(v_i) - 1 \geq n - d(v_2) - 1 \geq n - n/2 - 1 = n/2 - 1$, it has an independent set of size at least $n/6 - 1$. Let S_i' denote the subset of this independent set consisting only of vertices whose degrees do not exceed 10. We claim that $|S_i'| \geq n/15$. Indeed, otherwise there are at least $(n/6 - 1) - n/15 = n/10 - 1$ independent vertices of G with degree at least 11, which contradicts the fact that G has less than n edges. Next, we remove from S_i' any vertex which belongs to $N[B_1]$. Recall that $|B_1| = \delta$ and that each vertex of B_1 has degree at most $2\sqrt{n}$. Hence, $|N[B_1]| \leq 2\sqrt{n}\delta \leq 2\sqrt{n}\Delta \leq n/100$. It follows that $S_i = S_i' \setminus N[B_1] \geq n/15 - n/100 \geq n/18$ as required. $\qquad\square$

For $i = 2, \ldots, n$, let B_i be a random subset of S_i, where each vertex of S_i is independently selected to B_i with probability $n^{-1/2}$. Let

$$
\begin{aligned}
C_i &= (\cup_{j=2}^{i-1} B_j) \cap N(v_i) , \\
D_i &= B_i \setminus (\cup_{j=2}^{i-1} N[B_j]) .
\end{aligned}
$$

Clearly, $|C_i|$ could be as large as $d(v_i)$, which, in turn could be as large as $2n/i$. On the other hand, $|D_i|$ could be as small as zero. We will need, however, to make sure that $|C_i|$ is considerably smaller than $2n/i$, at least for relatively small i, and that $|D_i|$ is rather large, at least for relatively small i. This is made precise in the following lemma.

Lemma 2.2. *For n sufficiently large, all of the following hold with positive probability:*

1. *$|C_i| \leq 4\sqrt{n}$ for $i = 2, \ldots, n$,*

2. *$|D_i| \geq \sqrt{n}/50$ for $i = 2, \ldots, \lceil \sqrt{n}/10 \rceil$.*

Proof. We prove that each of the two sets of bounds hold with probability higher than $1/2$, and hence both hold with positive probability.

For the first part of the lemma, we only need to consider vertices v_i with $d(v_i) \geq 4\sqrt{n}$, as for other vertices the claim clearly holds since $|C_i| \leq |N(v_i)| = d(v_i)$. So, fix some vertex v_i with $d(v_i) \geq 4\sqrt{n}$. We prove that the probability that $|C_i| > 4\sqrt{n}$ is smaller than $1/(2n)$ and hence, by the union bound, this part of the lemma holds with probability greater than $1/2$. For $u \in N(v_i)$, the probability that $u \in B_j$ is at most $n^{-1/2}$ (it is either $n^{-1/2}$ if $u \in S_j$ or 0 if $u \notin S_j$). As the membership of u in C_i is only determined by its membership in $B_2 \cup \cdots \cup B_{i-1}$, we have that $\Pr[u \in C_i] \leq (i - 2)n^{-1/2}$. By Lemma 2.1, $i < 2n/d(v_i)$. Hence,

$$
\Pr[u \in C_i] \leq \frac{2\sqrt{n}}{d(v_i)} .
$$

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Observe that $|C_i|$ is a sum of $d(v_i)$ independent indicator random variables, one for each $u \in N(v_i)$, each variable having success probability at most $2\sqrt{n}/d(v_i)$. The expectation of $|C_i|$ is therefore at most $2\sqrt{n}$ and by a large deviation inequality of Chernoff (see [1], Theorem A.1.11), the probability of $|C_i|$ being larger than $4\sqrt{n}$ is exponentially small in $\sqrt{n}$. In particular, for n sufficiently large, it is smaller than $1/(2n)$.

For the second part of the lemma, observe that for $u \in S_i$, the probability that $u \in B_i$ is $n^{-1/2}$. On the other hand, for any $j \geq 2$, the probability that $u \notin N[B_j]$ is at least $1 - 10n^{-1/2}$. Indeed, this is true because either $u \in S_j$ in which case u is selected for B_j (and therefore for $N[B_j]$) with probability at most $1/\sqrt{n}$. Else, since $u \in S_i$ we already know that $d(u) \leq 10$. Hence, u has at most 10 neighbors in S_j so $u \in N[B_j]$ with probability at most $10/\sqrt{n}$. Hence, as long as $i \leq \lceil \sqrt{n}/10 \rceil$,

$$\Pr[u \in D_i] \geq n^{-1/2}(1 - 10n^{-1/2})^{i-2} \geq \frac{1}{e\sqrt{n}} \, .$$

Observe that $|D_i|$ is a sum of $|S_i|$ independent indicator random variables, each having success probability at least $\frac{1}{e\sqrt{n}}$. By Lemma 2.1, $|S_i| \geq n/18$, and therefore the expectation of $|D_i|$ is at least $\sqrt{n}/(18e) > \sqrt{n}/49$. By a large deviation inequality of Chernoff (see [1], Theorem A.1.13), the probability that $|D_i|$ falls below say, $\sqrt{n}/50$ is exponentially small in $\sqrt{n}$. In particular, for n sufficiently large, it is smaller than $1/(2n)$, so by the union bound, the second part of the lemma holds with probability greater than $1/2$ for all $i = 2, \ldots, \lceil \sqrt{n} \rceil$. □

Completing the proof of Theorem 1.1: By Lemma 2.2 we may fix independent sets $B_2, \ldots, B_n$ satisfying all the conditions of Lemma 2.2 with respect to the cardinalities of the sets C_i and D_i. The construction of the bijection f is done in four stages. At each point of the construction, some vertices of V are *matched* to some vertices of W while the other vertices of V and W are yet unmatched. Initially, all vertices are unmatched. We always maintain the *packing* property: for any two matched vertices $u, v \in V$ with $(u, v) \in E$, their corresponding matches $f(u)$ and $f(v)$ satisfy $(f(u), f(v)) \notin F$. Thus, once all vertices are matched, f is a packing of G and H.

Stage 1. We match v_1 (which, by definition, is a vertex with maximum degree in G) with a vertex $w \in W$ having *minimum* degree in H, and set $f(v_1) = w$. Let $N(w)$ be the set of neighbors of w in H. As $|N(w)| = \delta$ and since, by Lemma 2.1, $|S_1| \geq \delta$, we may match B_1 with $N(w)$ (recall that $B_1 \subset S_1$ is a set of size δ). Observe that the packing property is maintained since S_1 (and therefore B_1) is an independent set of non-neighbors of v_1. Note that after stage 1, precisely $\delta + 1$ pairs are matched.

Stage 2. Let k be the largest index such that $d(v_k) \geq 20\sqrt{n}$. Observe that by Lemma 2.1, $0 \leq k \leq \sqrt{n}/10$. This stage is done repeatedly for $i = 2, \ldots, k$, where at iteration i we match v_i and some subset of vertices of B_i with a corresponding set of vertices of W. Throughout this stage we maintain the following invariants:

1. After iteration i which matches v_i with some vertex $f(v_i)$, we also make sure that all neighbors of $f(v_i)$ in H are matched to vertices of B_i.

2. After iteration i, any matched vertex of V other than $v_1, \ldots, v_i$ is contained in $\cup_{j=1}^{i} B_j$.

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3. The overall number of matched vertices after iteration i is at most $i(\Delta(H)+1)$.

Observe that Stage 1 guarantees that these invariants hold at the beginning of Stage 2. Indeed, at the end of Stage 1, all the invariants hold for $i = 1$. In particular, recall that precisely $\delta+1$ vertices have been matched at Stage 1.

So, consider the i'th iteration of Stage 2, where v_i is some yet unmatched vertex with $d(v_i) \geq 20\sqrt{n}$. (Note that v_i is indeed yet unmatched as the vertices of each B_i, including B_1, have degree smaller than $2\sqrt{n}$.) We partition $N(v_i)$ into three parts $N(v_i) = X \cup Y \cup Z$, where X are the matched neighbors v_j with $j < i$, Y are the other matched neighbors, and Z being the yet unmatched neighbors. Clearly $|X| < i \leq k \leq \sqrt{n}/10$. On the other hand, by the second invariant, $Y \subset \cup_{j=1}^{i-1} B_j$. Thus, $Y \subset C_i \cup B_1$. From the first property in Lemma 2.2, together with $|B_1| = \delta$, we obtain that $|Y| \leq \delta + 4\sqrt{n} < 5\sqrt{n}$, and therefore $|X \cup Y| < 6\sqrt{n}$.

Consider the set T of $|X \cup Y|$ matches of $X \cup Y$ in H. Each vertex of T has at most $\Delta(H)$ neighbors in H, so altogether, there is a set Q with

$$|Q| \geq n - |T|\Delta(H) = n - |X \cup Y|\Delta(H) \geq n - 6\sqrt{n} \cdot \sqrt{n}/200 = 97n/100$$

vertices of H that are non-neighbors of all vertices of T. In order to maintain the packing property, we would like to match v_i with some vertex of Q. In order to do this, we must make sure that Q contains at least one vertex that is yet unmatched. This, in turn, is true because of the third invariant, as the overall number of matched vertices at this point is only $(i-1)(\Delta(H)+1) < k(\sqrt{n}/200 + 1) \leq n/2000 + \sqrt{n}/10$. So, the number of unmatched vertices is much larger than $n - |Q|$ and hence intersects Q. Let, therefore, $f(v_i)$ be one such vertex.

Let R be the set of neighbors of $f(v_i)$ in H that are still not matched. Clearly, $|R| \leq \Delta(H) \leq \sqrt{n}/200$. In order to maintain the first invariant, we must match some unmatched independent set of non-neighbors of v_i with R. A valid choice of such vertices which maintains the packing property is obtained by taking $|R|$ vertices of $D_i = B_i \setminus (\cup_{j=2}^{i-1} N[B_j])$, and this will also show that the second invariant is maintained (recall also that S_i, and hence B_i, and hence D_i, do not contain vertices of $N[B_1]$). We can, indeed, pick such a subset, as the second property in Lemma 2.2 guarantees that $|D_i| \geq \sqrt{n}/50 \geq |R|$. Finally, notice that the third invariant is maintained as iteration i only introduced $|R| + 1 \leq \Delta(H) + 1$ newly matched vertices.

Stage 3. At this point we are guaranteed that the unmatched vertices of G have degree less than $20\sqrt{n}$. Furthermore, by the third invariant of Stage 2, the total number of unmatched vertices of G is at least $n - (\sqrt{n}/10)(\Delta(H)+1) > 19n/20$. As the unmatched vertices of G induce a subgraph with at least $19n/20$ vertices and less than n edges, they contain an independent set of size at least $n/4$. Let, therefore, J denote a maximum independent set of unmatched vertices of G and let K be the remaining unmatched vertices of G. We have $|J| \geq n/4$.

The third stage consists of matching the vertices of K one by one. Suppose $v \in K$ is still unmatched. As $d(v) \leq 20\sqrt{n}$, the set X of matched neighbors of v satisfies $|X| \leq 20\sqrt{n}$. A similar argument to the one in Stage 2 now follows. Consider the set T of $|X|$ matches of X in H. Each vertex of T has at most $\Delta(H)$ neighbors in H, so altogether, there is a set Q of at least $n - |X|\Delta \geq n - n/10 = 9n/10$ vertices of H that are non-neighbors of all vertices of T. In order

to maintain the packing property, we would like to match v to some vertex of Q. In order to do this, we must make sure that Q contains at least one vertex that is yet unmatched. This, in turn, is true because the overall number of matched vertices at this point is at most $n - |J| \leq 3n/4$. So, there is a yet unmatched vertex of Q.

Stage 4. It remains to match the vertices of J to the remaining unmatched vertices of H, denoted by Q. Construct a bipartite graph P whose sides are J and Q. Recall that $|J| = |Q| \geq n/4$. We place an edge from $v \in J$ to $q \in Q$ if matching v to q is *allowed*. By this we mean that mapping v to q will not violate the packing property. At the beginning of Stage 4, as in Stage 3, for each $v \in J$, there are at least $9n/10$ vertices of H that are non-neighbors of all vertices that are matches of matched neighbors of v. So, the degree of v in P is at least $9n/10 - (n - |J|) > |J|/2$.

On the other hand, consider some $q \in Q$. Let T be the set of matched neighbors of q in H, and let X be their matches in G. Notice that q is not allowed to be matched to an unmatched neighbor of some $x \in X$. So, consider some $x \in X$. Clearly, if $d(x) \geq 20\sqrt{n}$ then all of the neighbors of x in G are already guaranteed to be matched by the first invariant in Stage 2. So, let $X' \subset X$ be the vertices with degrees smaller than $20\sqrt{n}$. Hence, q is not allowed to be matched to at most $|X'|20\sqrt{n}$ vertices of J, but $|X'| \leq |X| = |T| \leq \Delta(H)$ so q cannot be matched to at most $n/10$ vertices of J, which is smaller than $|J|/2$. Thus, the degree of q in P is also larger than $|J|/2$. It now follows by Hall's Theorem that P has a perfect matching, completing the matching f. $\qquad\square$

3   A counter-example for hypergraphs

In this short section we prove Proposition 1.2. This provides a counter-example to the conjecture of [5]. Indeed, the proposition gives a 3-graph H on n vertices whose set of links $\mathcal{L}$ contains a graph with one edge and hence satisfies $ex(n - 1, \mathcal{L}) = 0$. On the other hand, by the proposition, $ex(n, H) \geq \binom{n-1}{3} + \frac{1}{3}\binom{n-2}{2} > \binom{n-1}{3}$.

Proof of Proposition 1.2. Let T be the following 3-uniform hypergraph on the set of vertices $U \cup \{x, y\}$, where $|U| = n - 2$ and $x, y \notin U$. Let $U = U_1 \cup U_2 \cup U_3$ be a partition of U into 3 nearly equal disjoint sets (that is, each U_i is of cardinality either $\lfloor (n - 2)/3 \rfloor$ or $\lceil (n - 2)/3 \rceil$.) The edges of T consist of all 3-subsets of U, as well as all edges $\{x, u_i, u_j\}$ and $\{y, u_i, u_j\}$ with $1 \leq i < j \leq 3$, $u_i \in U_i$ and $u_j \in U_j$.

Note that T does not contain a copy of H, as it has 2 vertices (x and y) whose links are 3-colorable, and thus none of them lies in a copy of a complete 3-graph on 5 vertices. The desired result follows, as T has at least $\binom{n-2}{3} + \frac{4}{3}\binom{n-2}{2}$ edges. $\qquad\square$

Remark. The above reasoning can clearly provide many additional counter-examples. Indeed, any bounded degree 3-graph in which the links of all vertices but one are of chromatic number at least 4, and the last link is of chromatic number 2, is a counter-example. There are additional variants that provide more examples, but for all the ones we know, the reason that the extremal number is large is local, that is, one can construct a nearly complete hypergraph T containing a set of some f vertices that cannot serve as any set of f vertices of H, where f is bounded by a function of

$\Delta(H)$. It will be interesting to decide if for any bounded degree hypergraph H on n vertices with no isolated ones, the correct value of $ex(n, H)$ can be determined by ensuring that there is no such local obstruction.

4 Concluding remarks

We established a far-reaching generalization of Ore's Theorem, supplying the precise value of the extremal number $ex(n, H)$ for a large number of graphs H on n vertices. Note that the result shows that any graph on n vertices and more than $\binom{n-1}{2} + \delta - 1$ edges is *universal* for the class of all graphs with n vertices, minimum degree at most δ and maximum degree at most $\sqrt{n}/200$, that is, it contains all of them.

The extremal graph in Ore's Theorem is unique for all $n > 5$, that is, when $H = C_n$ is the cycle of length $n > 5$, the only graph on n vertices with $ex(n, H) = \binom{n-1}{2} + 1$ edges containing no copy of H is $K_n - S_{1,n-2}$, that is, the graph obtained from K_n by deleting a star with $n - 2$ edges. This is *not* the case in our more general Theorem 1.1. Indeed, let H be, for example, any bounded degree graph on n vertices in which all vertices but one have degree at least 3, and one vertex, call it v, is of degree 2 and its two neighbors x and y in H are adjacent. By Theorem 1.1, $ex(n, H) = \binom{n-1}{2} + 1$, and one graph on n vertices and $ex(n, H)$ edges containing no copy of H is $K_n - S_{1,n-2}$. There is, however, another extremal graph- the graph T obtained from K_n by deleting a vertex-disjoint union of a star with $n - 3$ edges and a single edge. Indeed, assuming H is a subgraph of such a graph, then the apex of the deleted star must play the role of v, but then its only two neighbors in T must play the roles of x and y, which is impossible, as x and y are not adjacent in T.

As mentioned in the remark following the proof of Proposition 1.2, all our counter-examples to the conjecture of [5] regarding the extremal numbers $ex(n, H)$ for hypergraphs H are based on a local obstruction. It seems interesting to decide if these are all the possible examples. One way to formalize this question is the following.

Question. Is it true that for any $k \geq 2$ and any $\Delta > 0$ there is an $f = f(\Delta)$ so that for any k-uniform hypergraph H on n vertices with no isolated vertices and with maximum degree at most Δ, any k-uniform hypergraph on n vertices which contains no copy of H and has the maximum possible number, $ex(n, H)$, of edges, must contain a complete k-hypergraph on at least $n - f$ vertices?

Our proof of Theorem 1.1 implies the validity of this assertion for graphs (that is, for the case $k = 2$). Indeed, if the maximum degree $d(v_1)$ of the graph G of missing edges considered in the proof satisfies, say, $n - d(v_1) - 1 \geq 6\delta$, then as shown in the proof, stage 1 can be completed. In all other stages we can allow more missing edges and the proof can still be carried out with no change. On the other hand, if $n - d(v_1) - 1 < 6\delta$, then the graph whose only missing edges are those of G contains a complete graph on more than $n - 12\delta$ vertices. Therefore, despite the fact that the extremal graph in the theorem is not unique, in general, as mentioned above, all extremal

examples must contain a clique on nearly all the vertices, and thus satisfy the statement in the previous question.

An equivalent formulation of Theorem 1.1 is in terms of graph packing. We have shown that for large n, any two graphs on n vertices, where one graph has minimum degree $\delta > 0$ and maximum degree at most $\sqrt{n}/200$ and the other has at most $n-1-\delta$ edges, pack. There is an extensive literature dealing with sufficient conditions ensuring that two graphs G and H on n vertices pack. The main open conjecture on the subject is the one of Bollobás and Eldridge [3] asserting that if $(\Delta(G)+1)(\Delta(H)+1) \leq n+1$ then G and H pack. Sauer and Spencer ([9], see also Catlin's paper [4]) proved that this is the case if $2\Delta(G)\Delta(H) < n$. For a survey of packing results including many extensions, variants and relevant references, see [7].

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