Piercing Convex Sets

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Abstract

A family of sets has the (p,q) *property* if among any p members of the family some q have a nonempty intersection. It is shown that for every $p \geq q \geq d+1$ there is a $c = c(p,q,d) < \infty$ such that for every family $\mathcal{F}$ of compact, convex sets in R^d which has the (p,q) property there is a set of at most c points in R^d that intersects each member of $\mathcal{F}$. This extends Helly's Theorem and settles an old problem of Hadwiger and Debrunner.

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1 Introduction

For two integers $p \geq q$, a family of sets $\mathcal{H}$ has the (p, q) *property* if among any p members of the family some q have a nonempty intersection. $\mathcal{H}$ is k*-pierceable* if it can be split into k or fewer subsets, each having a nonempty intersection. The *piercing number* of $\mathcal{H}$, denoted by $P(\mathcal{H})$, is the minimum value of k such that $\mathcal{H}$ is k-pierceable. (If no such finite k exists, then $P(\mathcal{H}) = \infty$.)

The classical theorem of Helly [13] states that any family of compact convex sets in R^d which satisfies the $(d+1, d+1)$-property is 1-pierceable. Hadwiger and Debrunner considered the more general problem of studying the piercing numbers of families $\mathcal{F}$ of compact, convex sets in R^d that satisfy the (p, q) property. By considering the intersections of hyperplanes in general position in R^d with an appropriate box one easily checks that for $q \leq d$ the piercing number can be infinite, even if $p = q$. Thus we may assume that $p \geq q \geq d + 1$.

Let $M(p, q; d)$ denote the maximum possible piercing number (which is possibly infinity) of a family of compact convex sets in R^d with the (p, q)-property. By Helly's Theorem,

$$M(d + 1, d + 1; d) = 1$$

for all d, and trivially $M(p, q; d) \geq p - q + 1$. Hadwiger and Debrunner [11] proved that for $p \geq q \geq d + 1$ that satisfy

$$p(d - 1) < (q - 1)d \tag{1}$$

this is tight, i.e., $M(p, q; d) = p - q + 1$. In all other cases, it is not even known if $M(p, q; d)$ is finite, and the question of deciding if this function is finite, raised by Hadwiger and Debrunner in 1957 in [11] remained open. This question, which is usually referred to as the (p, q)-problem, is considered in various survey articles and books, including [12], [4] and [7]. The smallest case in which finiteness is unknown, which is pointed out in all the above mentioned articles, is the special case $p = 4, q = 3, d = 2$. We note that in all the cases where finiteness is known, in fact $M(p, q; d) = p - q + 1$ and that there are examples of Danzer and Grünbaum (cf. [12]) that show that $M(4, 3; 2) \geq 3 > 4 - 3 + 1$.

The (p, q)-problem received a considerable amount of attention, and finiteness have been proved for various restricted classes of convex sets, including the family of parallelotopes with edges parallel to the coordinate axes in R^d ([12],[18], [5]), families of homothetes of a convex set ([18]), and, using a similar approach, families of convex sets with a certain "squareness" property ([8], see also [20]).

Despite these efforts, the problem of deciding if $M(p,q;d)$ is finite remained open for all values of $p \geq q \geq d+1$ which do not satisfy (1).

Here we solve this problem, by proving the following theorem.

Theorem 1.1 *For every $p \geq q \geq d+1$ there is a $c = c(p,q,d) < \infty$ such that $M(p,q;d) \leq c$. I.e., for every family $\mathcal{F}$ of compact, convex sets in R^d which has the (p,q) property there is a set of at most c points in R^d that intersects each member of $\mathcal{F}$.*

The detailed proof will appear in the full version of the paper. Here we briefly sketch the main ideas. Three tools are applied; a fractional version of Helly's Theorem, first proved in [14], Farkas' Lemma (or Linear Programming Duality) and a recent result proved in [1].

It may seem that there are almost no interesting families of compact convex sets in R^d which satisfy the (p,q)-property, for some $p \geq q \geq d+1$. A large class of examples can be constructed as follows. Let μ be an arbitrary probability distribution on R^d, and let $\mathcal{F}$ be the family of all compact convex sets F in R^d satisfying $\mu(F) \geq \epsilon$. Since the sum of the measures of any set of more than d/ϵ such sets is greater than d it follows that if p is the smallest integer strictly larger than d/ϵ then $\mathcal{F}$ has the $(p, d+1)$ property. It follows that $P(\mathcal{F}) \leq M(p, d+1; d)$, i.e., for every probability measure in R^d there is a set X of at most $M(p, d+1; d)$ points such that any compact convex set in R^d whose measure exceeds ϵ intersects X.

The following Theorem is an immediate consequence of Theorem 1.1.

Theorem 1.2 *Let $\mathcal{F}$ be a family of compact convex sets in R^d, and suppose that for every subfamily $\mathcal{F}'$ of cardinality x of $\mathcal{F}$ the inequality $P(\mathcal{F}') < \lceil x/d \rceil$ holds; i.e., $\mathcal{F}'$ can be pierced by less than x/d points. Then $P(\mathcal{F}) \leq M(x, d+1; d)$.*

Observe that in order to deduce a finite upper bound for the piercing number of $\mathcal{F}$, the assumption that $P(\mathcal{F}') < \lceil x/d \rceil$ cannot be replaced by $P(\mathcal{F}') \leq \lceil x/d \rceil$ as shown by an infinite family of hyperplanes in general position (intersected with an appropriate box), whose piercing number is infinite.

2 A sketch of the proofs

Since we do not try to optimize the constants here, and since obviously $M(p,q;d) \leq M(p,d+1;d)$ for all $p \geq q \geq d+1$ it suffices to prove an upper bound for $M(p,d+1;d)$. Another simple observation is that by compactness we can restrict our attention to finite families of convex sets.

Let $\mathcal{F}$ be a family of n convex sets in R^d, and suppose that $\mathcal{F}$ has the $(p,d+1)$ property. Our objective is to find an upper bound for the piercing number $P(\mathcal{F})$ of $\mathcal{F}$, where the bound depends only on p and d. It is convenient to describe the ideas in three subsections.

2.1 A fractional version of Helly's Theorem

Katchalski and Liu [14] proved the following result which can be viewd as a fractional version of Helly's Theorem.

Theorem 2.1 ([14]) *For every $0 < \alpha \leq 1$ and for every d there is a $\delta = \delta(\alpha, d) > 0$ such that for every $n \geq d+1$, every family of n convex sets in R^d which contains at least $\alpha\binom{n}{d+1}$ intersecting subfamilies of cardinality $d+1$ contains an intersecting subfamily of at least δn of the sets.*

Notice that Helly's Theorem is equivalent to the statement that in the above theorem $\delta(1,d) = 1$.

A sharp quantitative version of this theorem was proved by Kalai [15] and, independently, by Eckhoff [6]. See also [2] for a very short proof. All these proofs rely on Wegner's Theorem [19] that assrerts that the nerve of a family of convex sets in R^d is d-collapsible.

The above Theorem, together with a simple probabilistic argument, can be applied to prove the following lemma.

Lemma 2.2 *For every $p \geq d+1$ there is a positive constant $\beta = \beta(p,d)$ with the following property. Let $\mathcal{F} = \{A_1, \ldots, A_n\}$ be a family of n convex sets in R^d which has the $(p,d+1)$ property. Let a_i be nonnegative integers, define $m = \sum_{i=1}^{n} a_i$ and let $\mathcal{G}$ be the family of cardinality m consisting of a_i copies of A_i, for $1 \leq i \leq n$. Then there is a point x in R^d that belongs to at least βm members of $\mathcal{G}$.*

2.2 Farkas' Lemma and a Lemma on Hypergraphs

The following is a known variant of the well known lemma of Farkas (cf. [16], page 90).

Lemma 2.3 *Let A be a real matrix and b a real (column) vector. Then the system $Ax \leq b$ has a solution $x \geq 0$ if and only if for every (row) vector $y \geq 0$ which satisfies $yA \geq 0$ the inequality $yb \geq 0$ holds.*

This lemma (or the MinMax Theorem) can be used to prove the following.

Corollary 2.4 *Let $H = (V, E)$ be a hypergraph and let $0 \leq \gamma \leq 1$ be a real. Then the following two conditions are equivalent.*
(i) There exists a weight function $f : V \mapsto R^+$ satisfying $\sum_{v \in V} f(v) = 1$ and $\sum_{v \in e} f(v) \geq \gamma$ for all $e \in E$.
(ii) For every function $g : E \mapsto R^+$ there is a vertex $v \in V$ such that $\sum_{e;\ v \in e} g(e) \geq \gamma \sum_{e \in E} g(e)$.

By the last corollary and Lemma 2.2 one can prove the following result.

Corollary 2.5 *Suppose $p \geq d + 1$ and let $\beta = \beta(p, d)$ be the constant from Lemma 2.2. Then for every family $\mathcal{F} = \{A_1, \ldots, A_n\}$ of n convex sets in R^d with the $(p, d + 1)$ property there is a finite (multi)-set $Y \subset R^d$ such that $|Y \cap A_i| \geq \beta|Y|$ for all $1 \leq i \leq n$.*

2.3 Weak ϵ-nets for convex sets

The follwing result is proved in [1].

Theorem 2.6 ([1]) *For every real $0 < \epsilon < 1$ and for every integer d there exists a constant $b = b(\epsilon, d)$ such that the following holds.*
For every m and for every multiset Y of m points in R^d, there is a subset X of at most b points in R^d such that the convex hull of any subset of ϵm members of Y contains at least one point of X.

Several arguments that supply various upper bounds for $b(\epsilon, d)$ are given in [1]. The simplest one is based on a result of Bárány [3] whose proof is based on a deep result of Tverberg [17].
Theorem 1.1 follows from the above results quite easily. Let $\mathcal{F} = \{A_1, \ldots, A_n\}$ be a family of n convex sets in R^d with the $(p, d + 1)$ property , where $p \geq d + 1$. By Corollary 2.5 there is a finite (multi)-set $Y \subset R^d$ such that $|Y \cap A_i| \geq \beta|Y|$ for all $1 \leq i \leq n$, where $\beta = \beta(p, d)$ is as in Lemma 2.2. By Theorem 2.6 there is a set X of at most $b(\beta, d)$ points in R^d such that the convex hull of any set of $\beta|Y|$ members of Y contains at least one point of X. Since each member of $\mathcal{F}$ contains at least

$\beta|Y|$ points in Y it must contain at least one point of X. Therefore, $P(\mathcal{F}) \leq |X| \leq b(\beta(p,d),d)$, completing the proof. □

The detailed proofs of the lemmas and corollaries above, as well as some methods to improve the estimates for the numbers $M(p,q;d)$ using the known results about Turán's problem for hypergraphs together with some of the ideas of [1], will appear in the full version of the paper. The problem of determining the numbers $M(p,q;d)$ precisely for all $p \geq q \geq d+1$ remains wide open.

References

[1] N. Alon, I. Bárány, Z. Füredi and D. J. Kleitman, *Point selections and weak ϵ-nets for convex hulls*, submitted.

[2] N. Alon and G. Kalai, *A simple proof of the upper bound theorem*, European J. Combinatorics 6 (1985), 211-214.

[3] I. Bárány, A generalization of Caratheodory's theorem, Discrete Math. 40 (1982), 141-152.

[4] L. Danzer, B. Grünbaum and V. Klee, *Helly's theorem and its relatives*, Proc. Symp. Pure Math. 7 (Convexity), 101-180; Amer. Math. Soc., Providence, R.I., 1963.

[5] V. L. Dol'nikov, *A coloring problem*, (in Russian), Siberian Math. J. 13 (1972), 886-894; translated from Sibirsk. Mat. Zh. 13 (1972), 1272-1283.

[6] J. Eckhoff, *An upper bound theorem for families of convex sets*, Geom. Dedicata 19 (1985), 217-227.

[7] J. Eckhoff, *Helly, Radon, and Carathèodory type theorems*, to appear in: *Handbook of Convex Geometry.*

[8] B. E. Fullbright, *Intersectional properties of certain families of compact convex sets*, Pacific J. Math. 50 (1974), 57-62.

[9] B. Grünbaum, *On intersections of similar sets*, Portugal Math. 18 (1959), 155-164.

[10] B. Grünbaum, *Lectures on Combinatorial Geometry*, Mimeographed Notes, University of Washington, Seattle, 1974.

[11] H. Hadwiger and H. Debrunner, *Über eine Variante zum Helly'schen Satz*, Arch. Math. 8 (1957), 309-313.

[12] H. Hadwiger, H. Debrunner and V. Klee, *Combinatorial Geometry in the Plane*, Holt, Rinehart and Winston, New York, 1964.

[13] E. Helly, *Über Mengen konvexer Körper mit gemeinschaftlichen Punkten*, Jahresber. Deutsch. Math. Verein. 32 (1923), 175-176.

[14] M. Katchalski and A. Liu, *A problem of geometry in* R^d, Proc. Amer. Math. Soc. 75 (1979), 284-288.

[15] G. Kalai, *Intersection patterns of convex sets*, Israel J. Math. 48 (1984), 161-174.

[16] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, New York, 1986.

[17] H. Tverberg, *A generalization of Radon's Theorem*, J. London Math. Soc. 41 (1966), 123-128.

[18] G. Wegner, *Über eine kombinatorisch-geometrische Frage von Hadwiger und Debrunner*, Israel J. Math. 3 (1965), 187-198.

[19] G. Wegner, *d-collapsing and nerves of families of convex sets*, Arch. Math. 26 (1975), 317-321.

[20] G. Wegner, *Über Helly-Gallaische Stichzahlprobleme,* 3, Koll. Discrete Geom. Salzburg (1985), 277-282.