Partitioning all k-subsets into r-wise intersecting families

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Abstract

Let $r \geq 2$, n and k be integers satisfying $k \leq \frac{r-1}{r}n$. We conjecture that the family of all k-subsets of an n-set cannot be partitioned into fewer than $\lceil n - \frac{r}{r-1}(k-1) \rceil$ r-wise intersecting families. If true this is tight for all values of the parameters. The case $r = 2$ is Kneser's conjecture, proved by Lovász. Here we observe that the assertion also holds provided r is either a prime number or a power of 2.

1 Introduction

One of the earliest and possibly the best known application of topological methods in extremal combinatorics is the Kneser conjecture, now Lovász' Theorem [5]. This theorem asserts that for every $n \geq 2k$ it is impossible to split the family of all k-subsets of an n-set into fewer than $n - 2k + 2$ intersecting families. The main purpose of this brief note is to suggest the following conjecture, extending this result. Call a family of subsets r-wise intersecting if any collection of at most r subsets in it has a common point.

Conjecture 1.1. *Let $r \geq 2, n$ and k be integers, and suppose $k \leq \frac{r-1}{r}n$. Then the family of all k-subsets of $[n] = \{1, 2, \ldots, n\}$ cannot be partitioned into fewer than $\lceil n - \frac{r}{r-1}(k-1) \rceil$ r-wise intersecting families. This is tight for all admissible values of the parameters.*

The case $r = 2$ is Kneser's Conjecture proved by Lovász.

In this note we observe that the assertion of this conjecture holds for every prime r and for every r which is a power of 2. This is stated in the next theorem. It will be interesting to prove (or disprove) the statement for all the remaining values of r.

Theorem 1.2. *Let r be a prime or a power of 2, suppose $k \leq \frac{r-1}{r}n$ and let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ be a partition of all k-subsets of the set $[n] = \{1, 2, \ldots, n\}$ into m families, where each $\mathcal{F}_i$*

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is r-wise intersecting. Then $m \geq \lceil n - \frac{r}{r-1}(k-1) \rceil$. This is tight for all admissible values of the parameters.

2 Proof

One can prove Theorem 1.2 using tools from equivariant topology. It is much simpler, however, to deduce it from known results about the chromatic numbers of stable Kneser hypergraphs. We start by describing these results.

For $n \geq rk$ let $KG^r(k,n)$ denote the Kneser hypergraph whose vertex set is the set of all $\binom{n}{k}$ subsets of cardinality k of an n-set $[n] = \{1, 2, \ldots, n\}$ and whose edges are all r-tuples of k-sets which are pairwise disjoint. For $r = 2$ this is the Kneser graph. Lovász proved that the chromatic number of $KG^2(k,n)$ is $n - 2k + 2$ for all $n \geq 2k$, and in [3] it is proved that the chromatic number of $KG^r(k,n)$ is $\lceil \frac{n-r(k-1)}{r-1} \rceil$ for all $n \geq rk$.

Call a subset F of $[n]$ s-stable if any two elements of F are at distance at least s in the cyclic order on $[n]$. Let $KG^r(k,n)_{s-stab}$ be the induced sub-hypergraph of $KG^r(k,n)$ on the set of vertices which are s-stable. Ziegler [8], and Drewnowski, Łuczak and the present author [2] conjectured that the chromatic number of $KG^r(k,n)_{r-stab}$ is also $\lceil \frac{n-r(k-1)}{r-1} \rceil$, just as that of the full hypergraph $KG^r(k,n)$. We need the following known result regarding this conjecture.

Theorem 2.1 ([7], [2]). *If r is any power of 2 then the chromatic number of $KG^r(k,n)_{r-stab}$ is $\lceil \frac{n-r(k-1)}{r-1} \rceil$ for all admissible values of k, n.*

The case $r = 2$ was proved by Schrijver in [7], and in [2] it is shown that if the result holds for r_1 and r_2 then it also holds for $r_1 r_2$, implying the assertion for all powers of 2.

Improving results of Meunier [6] and of Alishahi and Hajiabolhassan [1], Frick proved in [4] that the chromatic number of $KG^r(k,n)_{2-stab}$ is $\lceil \frac{n-r(k-1)}{r-1} \rceil$. Another result proved in [4] is (a slightly stronger version of) the following.

Theorem 2.2 ([4], Theorem 3.10). *For any prime r and for any partition of $[n]$ into subsets C_i, each of size at most $r-1$, the induced subhypergraph of $KG^r(k,n)$ on the set of all vertices F that contain at most 1 element of each C_i has chromatic number $\lceil \frac{n-r(k-1)}{r-1} \rceil$.*

Proof of Theorem 1.2: The upper bound showing that the result is tight (for all values of r, if true) is simple. Put $s = \lfloor \frac{rk-1}{r-1} \rfloor$. Then every collection of r subsets of cardinality k of the subset $S = \{n-s+1, n-s+2, \ldots, n\}$ has a common point. For every $i \leq n-s$ let F_i be the collection of all k-subsets of $[n]$ whose minimum element is i. Let F_{n-s+1}

be the remaining k-subsets, that is, all those contained in S. This shows that there is a construction with

$$m = n - s + 1 = n - \lfloor \frac{rk-1}{r-1} \rfloor + 1 = \lceil n - \frac{r}{r-1}(k-1) \rceil,$$

as needed.

To prove the lower bound let n, k, r and $\mathcal{F}_1, \ldots \mathcal{F}_m$ be as in the statement of the theorem. Assume, first, that r is a prime. Let $C_1, C_2, \ldots, C_n$ be n pairwise disjoint sets, each of size $r-1$, and let $C = \cup_{i=1}^n C_i$ be their union. For each of the families $\mathcal{F}_i$, let $\mathcal{G}_i$ be the family of all k-subsets of C obtained as follows. For each member $F = \{i_1, i_2, \ldots, i_k\}$ of $\mathcal{F}_i$, let $C(F)$ denote the family of all $(r-1)^k$ subsets of C containing exactly one element of each C_{i_j} for $1 \leq j \leq k$. The family $\mathcal{G}_i$ is the union of all families $C(F)$ for $F \in \mathcal{F}_i$.

We claim that no set $\mathcal{G}_i$ contains r pairwise disjoint sets. Indeed, every collection of r (not necessarily distinct) members $G_1, G_2, \ldots, G_r$ of $\mathcal{G}_i$ consists of subsets that belong to $C(F_1), C(F_2), \ldots, C(F_r)$, respectively, for some (not necessarily distinct) members $F_j \in \mathcal{F}_i$. Since $\mathcal{F}_i$ is r-wise intersecting there is a common point, say ℓ, in all sets F_j. Thus each G_j contains a point of C_ℓ and as $|C_\ell| = r-1$ some pair of sets G_j intersect inside C_ℓ, by the pigeonhole principle. This proves the claim.

Note that the union of all families $\mathcal{G}_i$ is exactly the collection of all the k-subsets of C that contain at most 1 element from each C_i. Therefore the families $\mathcal{G}_i$ provide a proper coloring of the hypergraph described in Theorem 2.2 with parameters $(r-1)n$, r and k. The chromatic number of this hypergraph is $\lceil \frac{(r-1)n - r(k-1)}{r-1} \rceil$, providing the required lower bound for m.

The proof for r which is a power of 2 is similar, using the result in Theorem 2.1. We apply the same construction with sets C_i and families $\mathcal{G}_i$ as before, and place the sets C_i along a cycle of length $(r-1)n$, where each set C_i appears contiguously along the cycle. It is then easy to see that the union of the families $\mathcal{G}_i$ contains a copy of the hypergraph $KG^r((r-1)n, k)_{r-stab}$ on the set of vertices $C = \cup_{i=1}^n C_i$ of the cycle (as well as some additional hyperedges). As before, here too each family $\mathcal{G}_i$ contains no r pairwise disjoint members. The lower bound thus follows from Theorem 2.1. This completes the proof. Note that the proof shows that if the conjecture of [8] and [2] holds for all r then so does Conjecture 1.1. $\square$

Acknowledgment I thank Ryan Alweiss for many helpful discussions.

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