Every H-decomposition of K_n has a nearly resolvable alternative

Noga Alon * Raphael Yuster †

Abstract

Let H be a fixed graph. An *H-decomposition of* K_n is a coloring of the edges of K_n such that every color class forms a copy of H. Each copy is called a *member* of the decomposition. The *resolution number* of an H-decomposition L of K_n, denoted $\chi(L)$, is the minimum number t such that the color classes (i.e. the members) of L can be partitioned into t subsets $L_1, \ldots, L_t$, where any two members belonging to the same subset are vertex-disjoint. A trivial lower bound is $\chi(L) \geq \frac{n-1}{\overline{d}}$ where $\overline{d}$ is the average degree of H. We prove that whenever K_n has an H-decomposition, it also has a decomposition L satisfying $\chi(L) = \frac{n-1}{\overline{d}}(1 + o_n(1))$.

1 Introduction

All graphs and hypergraphs considered here are finite, undirected, simple, and have no isolated vertices. For standard graph-theoretic terminology the reader is referred to [2]. Let H and G be two graphs. An *H-decomposition* of G is a coloring of the edges of G, where each color class forms a copy of H. Each copy is called a *member* of the decomposition. An H-decomposition of K_n is called an *H-design* of n elements. H-designs are central objects in the area of Design Theory (cf. [3] for numerous results and references).

For a graph H, let $v(H)$, $e(H)$ and $gcd(H)$ denote, respectively, the number of vertices, the number of edges and the greatest common divisor of the degree sequence of H. If there is an H-decomposition of K_n, then, trivially, $e(H)$ divides $\binom{n}{2}$ and $gcd(H)$ divides $n-1$. In fact, Wilson has proved in a seminal result appearing in [9], that for every fixed graph H, if n is sufficiently large then these necessary conditions are also sufficient for the existence of an H-decomposition of K_n. For the rest of this paper we shall assume that these two necessary divisibility conditions hold.

Let L be an H-decomposition of K_n. The *resolution number of* L, denoted $\chi(L)$, is the minimum number t such that the members of L can be partitioned into t subsets $L_1, \ldots, L_t$, where any two members of L belonging to the same subset are vertex-disjoint. The use of χ follows from the

*Department of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel. Research supported in part by a USA-Israeli BSF grant, by the Israel Science Foundation and by the Hermann Minkowski Minerva Center for Geometry at Tel Aviv University. Email: noga@math.tau.ac.il

†Department of Mathematics, University of Haifa-ORANIM, Tivon 36006, Israel. e-mail: raphy@math.tau.ac.il

obvious fact that $\chi(L)$ is the chromatic number of the intersection graph of the members of L. The *resolution number of* H, denoted $\chi(H,n)$, is the minimum possible value of $\chi(L)$, ranging over all H-decompositions of K_n. Trivially, $\chi(H,n) \geq \frac{(n-1)v(H)}{2e(H)}$ since in any decomposition, the average number of members containing a vertex of K_n is precisely $\frac{(n-1)v(H)}{2e(H)}$. We say that K_n has a *resolvable H-decomposition* (also known as a *resolvable H-design*) if $\chi(H,n) = \frac{(n-1)v(H)}{2e(H)}$. There may be many distinct H-decompositions of K_n, and these decompositions may vary significantly in their properties. Some may be far from being resolvable. However, it has been proved by Ray-Chaudhuri and Wilson [7] that if $H = K_k$, n is a sufficiently large integer divisible by k, and $n-1$ is divisible by $k-1$, then there exists a resolvable K_k-decomposition of K_n. Namely, the $\binom{n}{2}/\binom{k}{2}$ members of the decomposition can be partitioned into $(n-1)/(k-1)$ subsets, where each subset consists of n/k vertex-disjoint copies of K_k (such a subset is called a *K_k-factor* of K_n). On the other hand, there are also non-resolvable K_k-decompositions of K_n. Some explicit constructions can be found in [3]. If H is an arbitrary graph, there is no analog to the Ray-Chaudhuri-Wilson theorem (namely, that the existence of the necessary conditions guarantees a resolvable H-decomposition of K_n for n sufficiently large). In fact, it is not difficult to show that for some graphs H no resolvable H-decomposition exists. Examples, which are easily verified, are $H = K_{1,t}$ where $t \geq 3$ is odd.

Our main result is that the obvious lower bound for $\chi(H,n)$ is asymptotically tight for every H.

Theorem 1.1 *Let H be a fixed graph with h vertices and m edges. Then,*

$$\chi(H,n) = (n-1)\frac{h}{2m}(1+o_n(1)).$$

As mentioned above, we cannot omit the error term completely, for general graphs H. The $o_n(1)$ error term in our proof of Theorem 1.1 is, in fact, a power n^β of n, where $\beta = \beta(H)$ is strictly less than 0.

Although in every H-decomposition there are at least $(n-1)\frac{h}{2m}$ members sharing a common vertex, we are able to prove that, for infinitely many n, there are H-decompositions of K_n in which *every* vertex appears in exactly $(n-1)\frac{h}{2m}$ members (note that this claim is interesting only if H is non-regular). Furthermore, for any two vertices of K_n, the number of members containing both of them is bounded. This is an easy corollary of the following theorem:

Theorem 1.2 *There exists a universal constant C such that if H is a fixed graph with h vertices and m edges and $x' \geq C \cdot \min\{m^{4/3}, h^2\}$ then there exists a nonempty regular graph G on x' vertices, which has an H-decomposition L with the property that every vertex of G appears in the same number of members of L.*

By applying Wilson's Theorem to G we get a G-decomposition of K_n. We now H-decompose each G so that the properties of Theorem 1.2 hold. This results in an H-decomposition of K_n

in which *every* vertex appears in exactly $(n-1)\frac{h}{2m}$ members. By applying Wilson's theorem to a complete graph K_k which, in turn, is G-decomposable, we get a K_k-decomposition of K_n. We now G-decompose each K_k, and H-decompose each resulting G, and obtain an H-decomposition of K_n which also has the additional property that any two vertices of K_n appear together in $O(k)$ members.

2 Proof of the main result

The proof of Theorem 1.1 requires a few lemmas. We first show that if K_n is H-decomposable, then it can also be decomposed into H-decomposable cliques whose sizes are bounded. Let F be a (possibly infinite) family of integers. Let $gcd(F)$ denote the largest positive integer which divides each number in F. Let $F_1 = \{n-1 \mid n \in F\}$ and let $F_2 = \{n(n-1)/2 \mid n \in F\}$. A *pairwise balanced design* is a partition of the complete graph into cliques (also called blocks). In [10] Wilson has proved the following:

Lemma 2.1 (Wilson [10]) *Let F be a finite family of positive integers. Then, there exists $n_0 = n_0(F)$ such that if $n > n_0$, $gcd(F_1)$ divides $n-1$ and $gcd(F_2)$ divides $\binom{n}{2}$ then there exists a pairwise balanced design of K_n, such that the size of each block belongs to F.* □

Let H be a graph, and let $T = \{n \mid K_n \text{ is } H\text{-decomposable}\}$. T is infinite but, obviously, $gcd(T_1)$ and $gcd(T_2)$ are finite. Thus, there are finite subsets $T^\alpha \subset T$ and $T^\beta \subset T$ such that $\gcd(T_1^\alpha) = gcd(T_1)$ and $\gcd(T_2^\beta) = gcd(T_2)$. Putting $F = T^\alpha \cup T^\beta$ yields a finite set of positive integers such that if $k \in F$ then K_k is H-decomposable, and if K_n is H-decomposable then $gcd(F_1)$ divides $n-1$ and $gcd(F_2)$ divides $\binom{n}{2}$. Applying Lemma 2.1 to this F we get:

Corollary 2.2 *For every graph H there is a finite set of positive integers $F = F(H)$ and a positive integer $N_1 = N_1(H)$, such that if $n > N_1$ and K_n is H-decomposable, then K_n is also decomposable into H-decomposable cliques whose sizes belong to F.* □

Recall that an h-uniform hypergraph is a collection of h-sets (the edges) of some n-set (the vertices). The degree $deg(x)$ of a vertex x in a hypergraph is the number of edges containing x. A *matching* in a hypergraph is a set of pairwise disjoint edges. The *chromatic index* of a hypergraph S, denoted $q(S)$, is the smallest integer q such that the set of edges of S can be partitioned into q matchings. A powerful theorem of Pippenger and Spencer [6] gives an asymptotically sharp estimate of $q(S)$ when S is a uniform hypergraph in which any two vertices appear together in a small number of edges. Better estimates for the error term have been proved subsequently in [4], [5]. Here we state the theorem in a slightly weaker form which suffices for our purposes.

Lemma 2.3 ([6], [4], [5]) *Let h and C be positive integers and let $\alpha < 1$ and $\epsilon < 1$ be positive real numbers. There exist $N_0 = N_0(h, C, \alpha, \epsilon)$ and $0 < \beta = \beta(h, C, \alpha, \epsilon) < 1$ such that the following holds: If S is an h-uniform hypergraph with $n > N_0$ vertices and:*

1. *There exists* $d > \epsilon n$ *such that for every* $x \in S$, $|deg(x) - d| < d^{\alpha}$.

2. *Any two vertices appear together in at most* C *edges.*

Then, $q(S) \le d + d^{\beta}$. $\square$

Let H have h vertices and m edges. Every H-decomposition of K_n defines an n-vertex h-uniform hypergraph whose edges correspond to the vertices of each member of the decomposition. Clearly, the chromatic index of this hypergraph is exactly the resolution number of the decomposition. It is our goal to show that there always exists an H-decomposition of K_n whose corresponding hypergraph satisfies the conditions of Lemma 2.3 with $\alpha = 0.6$, $C = C(H)$, $\epsilon = \epsilon(H)$ and $d = (n-1)\frac{h}{2m}$. For this purpose, we need the following simple lemma.

Lemma 2.4 *For every* $a > 0$ *there exists a* $T = T(a)$ *such that if* $t > T$ *and* $X_1, \ldots, X_t$ *are* t *mutually independent discrete random variables taking values between* 0 *and* a, *and* μ *is the expectation of* $X = X_1 + \ldots + X_t$ *then*

$$Pr[|X - \mu| > t^{0.51}] < \frac{1}{t^2}.$$

Proof: Several (related) proofs relying on some standard known bounds for large deviations can be given. Here we describe one that follows from Theorem 4.2 on page 90 of [1]. Let A be the set of reals, and put $B = \{1, 2, \ldots, t\}$. Let $g : B \mapsto A$ be a random function obtained by defining $g(i) = X_i/a$ for each $i \in B$. Define $B_i = \{1, 2, \ldots, i\}$ and put $L(g) = \sum_{i \in B} g(i)$ ($= X/a$). Notice that if g and g' differ only on $B_{i+1} - B_i$, then $|L(g) - L(g')| \le 1$. Therefore, by Theorem 4.2 on page 90 of [1], for every $\lambda > 0$,

$$Pr[|L(g) - E(L(g))| \ge \lambda\sqrt{t}] < 2e^{-\lambda^2/2}.$$

Since here $L(g) = X/a$ and $E(L(g)) = \mu/a$, the desired result follows by taking $\lambda = t^{0.01}/a$, and a sufficiently large T. $\square$

Proof of Theorem 1.1 Fix F and N_1 as in corollary 2.2. Now, define $C = \lfloor \frac{k-1}{\delta(H)} \rfloor$ where $k = k(H)$ is the largest integer in F, and $\delta(H)$ is the minimum degree of H. Define $\epsilon = \frac{h}{3m}$ and note that $\epsilon < 1$. Define $\alpha = 0.6$. Let $\beta = \beta(h, C, \alpha, \epsilon)$ and $N_0 = N_0(h, C, \alpha, \epsilon)$ be defined as in Lemma 2.3. For each $f \in F$, let L_f be a fixed H-decomposition of K_f. Picking a random vertex of K_f, let Y_f denote the random variable corresponding to the number of members of L_f containing the randomly selected vertex. Note that, trivially, each Y_f attains values between 1 and $f - 1 \le k - 1$. Let T be defined as in Lemma 2.4, applied to the constant $a = k - 1$. Finally, define

$$N = \max\{N_0\ ,\ N_1\ ,\ T(k-1)\ ,\ \left(\frac{2m}{h}\right)^{10}\ ,\ 2k^2\}.$$

We show that if $n > N$ and K_n is H-decomposable then $\chi(H, n) \le d + d^{\beta}$ where $d = (n-1)\frac{h}{2m}$. This will establish Theorem 1.1. Assume, therefore, that K_n is H-decomposable. Since $n > N_1$

we know, by Corollary 2.2, that K_n is also decomposable to H-decomposable cliques whose sizes belong to F. Let L^* be such a clique decomposition, and let $Q \in L^*$. Since Q is a clique isomorphic to some K_f, there are $f!$ different ways to decompose Q to copies of H using L_f, each corresponding to a permutation of the vertices of Q. For each $Q \in L^*$ we randomly choose such a permutation. All the $|L^*|$ choices are independent, and each choice is done according to a uniform distribution. Combining all these $|L^*|$ random H-decompositions, we obtain an H-decomposition L of K_n.

Claim 1: *With positive probability, each vertex of K_n appears in at least $d - d^\alpha$ members of L and in at most $d + d^\alpha$ members of L.*

Proof: Fix a vertex x of K_n. Let $deg(x)$ denote the number of members of L which contain x. Let $Q_1, \ldots, Q_t$ be the cliques of L^* which contain x, and let $f_1, \ldots, f_t$ be their corresponding sizes. Clearly, $f_1 + \ldots + f_t = n - 1 + t$. For $i = 1, \ldots, t$, let X_i be the number of members of L which contain x and whose edges belong to Q_i. Clearly, $\sum_{i=1}^t X_i = deg(x)$. Each X_i is a random variable whose expectation is exactly the average number of members of L_{f_i} which contain a vertex of K_{f_i}. Thus, $E[X_i] = (f_i - 1)\frac{h}{2m}$, and consequently

$$E[deg(x)] = \sum_{i=1}^{t}(f_i - 1)\frac{h}{2m} = (n-1)\frac{h}{2m} = d.$$

Furthermore, each X_i has the same distribution as the random variable Y_{f_i}, and $X_1, \ldots, X_t$ are independent. Since $t \geq (n-1)/(k-1) > T$ we have by Lemma 2.4 that:

$$\Pr\left[|deg(x) - d| > t^{0.51}\right] < \frac{1}{t^2}.$$

Since $n > N \geq (\frac{2m}{h})^{10}$ we have that $t^{0.51} < d^{0.6}$. Also note that $t^2 \geq (n-1)^2/(k-1)^2 > n$. Thus,

$$\Pr\left[|deg(x) - d| > d^{0.6}\right] < \frac{1}{n}.$$

Since there are n vertices to consider, it follows that with positive probability, for every vertex x of K_n, $|deg(x) - d| \leq d^{0.6} = d^\alpha$. $\Box$

Claim 2: *Any two vertices of K_n appear together in at most C members of L.*

Proof: If a member of L contains the vertices x and y, then the member belongs to the H-decomposition of the unique clique $X \in L^*$ which contains the edge (x, y). Since X has at most k vertices, there are at most $C = \lfloor (k-1)/\delta(H) \rfloor$ such members. $\Box$

Claims 1 and 2, together with the facts that $N > N_0$ and that $d > \epsilon n$ show that, with positive probability, the conditions of Lemma 2.3 are satisfied for the hypergraph corresponding to the decomposition L. Hence, there exists an H-decomposition L of K_n satisfying $\chi(L) \leq d + d^\beta$. $\Box$

Proof of Theorem 1.2: The vertices of H may be partitioned into two sets A and B where A consists of all vertices whose degree is at least $m^{1/3}$. Clearly, $|A| \leq 2m^{2/3}$. It is a well known theorem of Singer [8] that the abelian group Z_x has a subset S of $\Theta(\sqrt{x})$ elements such that all

possible differences (in Z_x) between any two elements of S, are distinct. We call S a *difference set*. Now let x be the smallest integer greater than $2m^{4/3}+h$ such that Z_x has a difference set of size $|A|$. Clearly, $x=\Theta(m^{4/3})$. We shall map the vertices of H to distinct elements of Z_x, such that if (a,b) and (c,d) are two distinct edges then $a-b \neq c-d \bmod x$ and $a-b \neq d-c \bmod x$. First, we map the vertices of A to some fixed difference set of Z_x having size $|A|$, using an arbitrary one to one mapping. Next, we assign values to the remaining vertices of B one by one, maintaining the required property. This is possible, since at each stage, the next vertex of B to be mapped, denoted y, should be connected to a subset T of at most $deg(y) < m^{1/3}$ already mapped vertices. Each vertex of T introduces at most $2z$ values to which y cannot be mapped, where $z \leq m$ is the number of edges of H connecting two previously mapped vertices. Altogether y cannot be mapped to at most $2z \cdot deg(y) < 2m^{4/3} \leq x-h$ elements of Z_x. Thus, there are at least h elements of Z_x to which y can be mapped. At least one of these elements is not assigned to a previously mapped vertex, so we map y to such an element.
We now consider a graph G whose vertices are the elements of Z_x. The edges are defined as follows. For each edge uv of H, let a and b be the elements of Z_x which were assigned to u and v respectively, in the mapping defined above. For $i=0,\ldots,x-1$, all the pairs $(a+i,b+i)$ are edges of G. It follows that G is $2m$-regular, and has an H-decomposition into x members. In fact, every vertex of G plays the role of each vertex of H exactly once. Clearly, for $x' > x$, the same arguments hold.

The bound $m^{4/3}$ in Theorem 1.2 can be replaced by the bound h^2 which results by mapping the vertices of H injectively into a difference set of size at least h in $Z_{x'}$. Such a set exists provided $x' \geq \Omega(h^2)$. □

3 Concluding remarks and open problems

1. As mentioned in the introduction, we cannot avoid an error term in the statement of Theorem 1.1, since there are graphs with no resolvable decomposition. The error term in the proof of Theorem 1.1 is $O(n^{\beta})$ for some $\beta < 1$. It is plausible, however, that the error term is bounded by a function of H. Namely,

   Conjecture 3.1
   $$\chi(H,n) \leq (n-1)\frac{h}{2m} + C(H).$$

2. Theorem 1.2 and the comment following it, show that for every graph H with h vertices and m edges, there is a *regular* graph G having $O(Min\{m^{4/3}\ ,\ h^2\})$ vertices which has an H-decomposition L in which every vertex is contained in the same number of members of L. Let us call such a decomposition a *regular decomposition*. We may now define $f(h,m)$ to be the smallest integer t such that for every graph H with h vertices and m edges, and for every $t' \geq t$, there are regular graphs with t' vertices which have a regular H-decomposition.

Similarly, we may restrict ourselves to some specific families of graphs, such as the family of trees, and define, for a family $\mathcal{F}$ of graphs, $f_{\mathcal{F}}(h,m)$ to be the smallest integer t such that for every graph $H \in \mathcal{F}$ with h vertices and m edges, and for every $t' \geq t$, there are regular graphs with t' vertices which have a regular H-decomposition.

Therefore, $f(h,m) = O(Min\{m^{4/3}\ ,\ h^2\})$. It is interesting to find more accurate upper and lower bounds for $f(h,m)$. It is not difficult to show that $f(h,m) = \Theta(h^2)$ when $m = \binom{h}{2} - 1$. A greedy algorithm shows that $f_{\mathcal{F}_d}(h,m) \leq (1+2d^2)h$ where $\mathcal{F}_d$ is the family of d-degenerate graphs. In particular, for trees we get $f_{\mathcal{F}_1}(h,h-1) \leq 3h$, while an easy lower bound, resulting from stars, is $2h-2$. It may be interesting to close this gap.

3. The main result of [6] easily implies that every K_k-decomposition of K_n is nearly resolvable, that is, for each such decomposition L, $\chi(L) = (1+o_n(1))\frac{n-1}{k-1}$. This is *not* the case for other graphs H. Thus, for example, it is easy to see that for the path of length 2, $H = K_{1,2}$, and for every n such that 2 divides $\binom{n}{2}$, there is an H-decomposition L of K_n in which $n-1$ members of L are incident with a single vertex, implying that $\chi(L) \geq n-1$ ($> \frac{3(n-1)}{4}$). Therefore, L is not nearly resolvable.

   Our main result here shows, however, that for every fixed graph H, even though there may be some H-decompositions of K_n which are not nearly resolvable, there always exist ones that are.

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