A note on the Running Intersection Property

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Abstract

A sequence of subsets $F_1, F_2, \ldots, F_m$ of a finite set X satisfies the **Running Intersection Property** (RIP) if for every $k > 1$ the intersection of F_k with the union of all previous F_j is contained in one of these previous subsets. A family of subsets $\mathcal{F}$ of X satisfies RIP^* if there is an ordering of its members that satisfies RIP.

A sequence of subsets $F_1, F_2, \ldots, F_m$ of a finite set X is **expansive** if for every $k > 1$, the cardinality of the intersection of F_k with the union of all previous sets is at least as large as the cardinality of the intersection of any subset that appears after F_k with this union. In this note we prove that if a family of subsets of a finite set satisfies RIP^*, then any expansive ordering of its members satisfies RIP. This settles a question of Spiegler.

1 The main result

A sequence of subsets $F_1, F_2, \ldots, F_m$ of a finite set X satisfies the **Running Intersection Property** (RIP) if for every $k > 1$ the intersection of F_k with the union of all previous F_j is contained in one of these previous subsets, that is,

$$\text{For every } k > 1 \text{ there is an } i < k \text{ so that } F_k \cap (\cup_{j<k} F_j) \subset F_i. \tag{1}$$

A family of subsets $\mathcal{F}$ of X satisfies RIP^* if there is an ordering of its members that satisfies RIP. For more on the Running Intersection Property see, e.g., [1], [2] and the references therein.

A sequence of subsets $F_1, F_2, \ldots, F_m$ of a finite set X is **expansive** if for every $k > 1$, the cardinality of the intersection of F_k with the union of all previous sets is at least as large as the cardinality of the intersection of any subset that appears after F_k with this union, that is,

$$\text{For every } k > 1, \; |F_k \cap (\cup_{j<k} F_j)| \geq |F_i \cap (\cup_{j<k} F_j)| \text{ for all } i > k. \tag{2}$$

R. Spiegler [3] conjectured that if a family of subsets satisfies RIP^*, then any expansive ordering of its members satisfies RIP. This is proved in the following theorem.

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Theorem 1.1 *Let $\mathcal{F}$ be a family of subsets of a finite set X. If $\mathcal{F}$ satisfies RIP^*, then any expansive ordering of its members satisfies RIP.*

Note that the above theorem supplies a simple efficient algorithm for checking if a given family $\mathcal{F}$ satisfies RIP^*: we simply produce an expansive ordering of it and check if it satisfies (1).

2 The proof

A family of subsets $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ of a finite set satisfies the **Tree Decomposition Property** (TDP) if there is a tree T on the set of vertices $\mathcal{F}$ so that for any $F_i, F_j, F_k \in \mathcal{F}$ where F_i is on the unique path in T from F_k to F_j, $F_k \cap F_j \subset F_i$. If this is the case we say that T, with an arbitrary vertex of it designated as a root, is a **realization for** $\mathcal{F}$. Note that there can be many trees realizing the same family $\mathcal{F}$.

An ordering of the vertices of a rooted tree is called **admissible** if the root appears first, and any other vertex appears after its unique parent in the tree. The following simple lemma appears, in various forms, in the literature, see, e.g., [1], Chapter 2. For completeness we include a short proof.

Lemma 2.1 *Let $\mathcal{F}$ be a family of subsets of a finite set X. Then $\mathcal{F}$ satisfies RIP^* if and only if it satisfies TDP. Moreover, if $\mathcal{F}$ satisfies TDP and T is a rooted tree realizing $\mathcal{F}$, then any admissible ordering of its members satisfies RIP.*

Proof: Assume first that $\mathcal{F}$ satisfies RIP^*. Then there is a sequence $F_1, F_2, \ldots, F_m$ of the members of $\mathcal{F}$ such that (1) holds. Let T be a rooted tree on the set of vertices $\mathcal{F}$, where F_1 is the root, and for each $k > 1$ the unique parent of F_k in the tree is an arbitrarily chosen F_i so that $i < k$ and

$$F_k \cap (\cup_{j<k} F_j) \subset F_i.$$

We claim that T is a realization for $\mathcal{F}$. Indeed, suppose $F_i, F_j, F_k \in \mathcal{F}$, with F_i being on the unique path between F_j and F_k. We have to show that $F_k \cap F_j \subset F_i$. This is proved by induction on the distance in T between F_k and F_j. If the distance is 0 or 1 there is nothing to prove as in this case F_i is either F_j or F_k. Assuming the assertion holds for distance smaller than d, suppose the distance between F_j and F_k is $d \geq 2$. Without loss of generality suppose that $j < k$. Let F_s be the unique parent of F_k in T. By the construction of T, $F_k \cap F_j \subset F_s$. It is also clear that F_s is on the unique path in T from F_k to F_j (since F_j is not a descendent of F_k). Thus either $s = i$, and then the desired result $F_k \cap F_j \subset F_i$ holds, or F_i is on the unique path in T between F_s and F_j. In the latter case, as the distance between F_s and F_j is $d - 1$ it follows, by the induction hypothesis, that $F_s \cap F_j \subset F_i$, completing the proof of the claim, as $F_k \cap F_j \subset F_s \cap F_j \subset F_i$.

Conversely, suppose that $\mathcal{F}$ satisfies TDP, and let T be a rooted tree which forms a realization for $\mathcal{F}$. Let F_1 be the root, and let $F_1, F_2, \ldots, F_m$ be an admissible order of the members of $\mathcal{F}$. We complete the proof of the lemma by showing that this ordering satisfies RIP. For $k > 1$, let F_i be the unique parent of F_k in T. Since the order is admissible $i < k$. In addition, F_i lies on the unique path between F_k and F_j for any $j < k$, as no such F_j is a descendent of F_k in T. As T satisfies TDP it follows that $F_k \cap F_j \subset F_i$ for each $j < k$, and hence $F_k \cap (\cup_{j<k} F_j) \subset F_i$, as needed. $\Box$

Proof of Theorem 1.1: Let $\mathcal{F}$ be a family of subsets of a finite set X and suppose it satisfies RIP^*. Let $F_1, F_2, \ldots, F_m$ be an expansive order of the members of $\mathcal{F}$. We have to show that this ordering satisfies RIP. To do so we prove the following:

Claim: For every $k \geq 1$ there is a tree that forms a realization for $\mathcal{F}$ so that $F_1, F_2, \ldots, F_k$ is an initial segment in an admissible ordering of the vertices of the tree.

Note that the case $k = m$ of the above lemma implies the assertion of the theorem, as it provides a realization for $\mathcal{F}$ in which the sequence $F_1, F_2, \ldots, F_m$ is admissible, and hence, by Lemma 2.1, this sequence satisfies RIP, as needed.

It remains to prove the claim. This is done by induction on k. The case $k = 1$ follows from Lemma 2.1. Assuming the assertion of the claim for $k - 1$, we prove it for k, $k \geq 2$.

By the induction hypothesis there is a tree T on the set of vertices $\mathcal{F}$ so that $F_1, F_2, \ldots, F_{k-1}$ is an initial segment in an admissible ordering of the vertices of the tree. Therefore, each F_j for $j \geq k$ is a descendent in T of at least one of the vertices in the set $\{F_1, F_2, \ldots, F_{k-1}\}$. In particular, this holds for F_k, let F_i be the first vertex in the path from F_k to the root F_1 in T so that $i \leq k - 1$. If F_i is the parent of F_k in T, then the tree T satisfies the assertion of the claim for k, establishing the required induction step. We thus assume that this is not the case and the path in T from F_i to F_k is the following: $F_i, G_1, G_2, \ldots, G_s, F_k$, where $G_j \in \{F_{k+1}, F_{k+2}, \ldots, F_m\}$ for all $j, 1 \leq j \leq s$.

Our objective is to transform T into another tree T' that satisfies the assertion of the claim for k. To this end we define several pieces of the tree T, as follows. Let T_0 be the subtree of T rooted at F_1 and consisting of all vertices of T besides G_1 and its descendents. Let T_1 denote the subtree of T rooted at G_1, besides G_2 and its descendents. Similarly, for each $q < s$, let T_q denote the subtree of T rooted at G_q besides G_{s+1} and its descendents. Let T_s be the subtree of T rooted at G_s besides F_k and its descendents. Finally, let T_∞ denote the subtree of T rooted at F_k.

Note that the tree T_0 contains all the vertices $F_1, F_2, \ldots, F_{k-1}$. This is because these vertices form an initial segment in an admissible ordering of the vertices of T, hence none of them can be a descendent of G_1, which is not in this initial segment.

Recall that $F_1, F_2, \ldots, F_m$ is an expansive ordering of the members of $\mathcal{F}$. Therefore,

$$|F_k \cap (\cup_{j<k} F_j)| \geq |G_q \cap (\cup_{j<k} F_j)| \tag{3}$$

for all $1 \leq q \leq s$. On the other hand, T is a realization for $\mathcal{F}$ which satisfies TDP, and as all vertices F_j for $j < k$ lie in T_0, it follows that for each such F_j, $F_k \cap F_j \subset F_i$ and also $F_k \cap F_i \subset G_q$ for all $1 \leq q \leq s$. We conclude that $F_k \cap (\cup_{j<k} F_j) = F_k \cap F_i$ is contained in $F_i \cap G_q$ for all $1 \leq q \leq s$, and by (3) we have

$$F_k \cap (\cup_{j<k} F_j) = F_k \cap F_i = F_i \cap G_1 = F_i \cap G_2 = \ldots = F_i \cap G_s.$$

We can now construct the tree T'. It is obtained from T by reversing the path between G_1 and F_k as follows: starting with the subtree T_0, connect to it the subtree T_∞ by making F_i the parent of

F_k. Next, connect the subtree T_s by making F_k the parent of G_s, the subtree T_{s-1} by making G_s the parent of G_{s-1} and so on until the tree T_1 which is connected by letting G_2 be the parent of G_1. Clearly $F_1, F_2, \ldots, F_k$ is an initial segment in an admissible ordering of T', and hence it only remains to check that the tree T' is indeed a realization of $\mathcal{F}$, namely, that for every three vertices along a path in the tree, the subset corresponding to the middle vertex is contained in the intersection of those corresponding to the other two subsets. This is obvious if all three vertices belong to T_0 or if none of them belongs to T_0. The only remaining cases are when the path is between a vertex in T_0 and a vertex not in T_0. In this case, the intersection of the corresponding sets is contained in the common value of $F_k \cap F_i = G_1 \cap F_i = G_2 \cap F_i = \ldots = G_s \cap F_i$ and the desired inclusion in T' follows from the corresponding one in T. This completes the proof of the claim, establishing the assertion of the theorem. $\square$

References

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