Coloring, sparseness, and girth

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Abstract

An *r-augmented tree* is a rooted tree plus r edges added from each leaf to ancestors. For $d, g, r \in \mathbb{N}$, we construct a bipartite r-augmented complete d-ary tree having girth at least g. The height of such trees must grow extremely rapidly in terms of the girth.

Using the resulting graphs, we construct sparse non-k-choosable bipartite graphs, showing that maximum average degree at most $2(k-1)$ is a sharp sufficient condition for k-choosability in bipartite graphs, even when requiring large girth. We also give a new simple construction of non-k-colorable graphs and hypergraphs with any girth g.

1   Introduction

A graph G is *k-choosable* if, for every way of assigning a list $L(v)$ of k colors to each vertex $v \in V(G)$, there is a proper coloring f of G with $f(v) \in L(v)$ for all v. The *choice number* of a graph is the least k such that it is k-choosable. If every subgraph has average degree less than k, then it has a vertex with degree less than k, and inductively it is k-choosable.

For bipartite graphs, one can guarantee k-choosability with average degree up to $2(k-1)$. Using (an early version of) the Combinatorial Nullstellensatz [1], Alon and Tarsi [2] proved Theorem 1.1 below, which implied the conjecture of [5] that planar bipartite graphs are 3-choosable. As mentioned in [2], another route to the result was subsequently noted by Bondy, Boppana, and Siegel, as follows. A *kernel* of a digraph is an independent set S containing

a successor of every vertex outside S. If a graph G has an orientation D with maximum outdegree less than k, and every induced subdigraph of D has a kernel, then inductively G is k-choosable. Richardson [14] proved that every digraph with no odd cycle has a kernel. Hakimi [7] proved that G has an orientation with maximum outdegree at most $k-1$ when all induced subgraphs have average degree at most $2(k-1)$.

Theorem 1.1 ([2]). *If G is a bipartite graph such that every subgraph has average degree at most $2(k-1)$, then G is k-choosable.*

We show that Theorem 1.1 is sharp in a strong sense: we construct non-k-choosable bipartite graphs G such that after deleting any edge from G, all subgraphs of the remaining graph have average degree at most $2(k-1)$. Thus our graphs are $(k+1)$-choice-critical. Furthermore, such examples exist with arbitrarily large girth. We prove the following theorem.

Theorem 1.2. *For $g, k \in \mathbb{N}$, there is a bipartite graph G with girth at least g that is not k-choosable even though every proper subgraph has average degree at most $2(k-1)$.*

To prove this, we consider a new problem. Let an *r-augmented tree* be a graph consisting of a rooted tree (called the *underlying tree*) plus edges from each leaf to r of its ancestors (called *augmenting edges*). A *complete d-ary tree of height m* is a rooted tree whose internal vertices have d children and whose leaves have distance m from the root. For $d, r, g \in \mathbb{N}$, let a *(d, r, g)-graph* be a bipartite r-augmented complete d-ary tree with girth at least g.

Theorem 1.3. *For $d, r, g \in \mathbb{N}$, there exists a (d, r, g)-graph.*

In Section 2 we prove Theorem 1.3, and in Section 3 we give several applications. In Section 3.1 we present a simple construction of t-uniform hypergraphs with arbitrarily large girth and chromatic number, for all t. For $t = 2$, Erdős [3] used the probabilistic method to prove existence; see also [4, 8] for subsequent work. Explicit constructions followed in [12, 13, 11]. These are inductive and, except for the last one, use hypergraphs with large edges. Using (d, r, g)-graphs (built inductively), our construction is non-inductive and does not involve hypergraphs with larger edges. Moreover, the same method provides explicit high girth hypergraphs of any uniformity based on (d, r, g)-graphs, without using hypergraphs (besides those constructed) in the process. One can also ask for the minimum number of edges in a k-critical n-vertex graph with girth at least g; such questions were studied in [8, 9].

We prove Theorem 1.2 in Section 3.2. Stronger versions involving restricted list assignments are proved in Section 3.3. For example, when the lists at adjacent vertices are disjoint, every coloring chosen from the lists is proper. We extend the analysis of the graph constructed for Theorem 1.2 by constructing a k-list assignment in which any two adjacent lists have exactly one common color and yet no proper coloring can be chosen.

One can also restrict list assignments by bounding the size of the union of the lists. For bipartite graphs, a proper coloring can be chosen from any k-lists whose union has size at most $2k - 2$. We prove that this is sharp (for any girth) by constructing a bipartite graph with k-lists whose union has size $2k - 1$ from which no proper coloring can be chosen.

Finally, in Section 4 we discuss the height of the trees used in Theorem 1.3. For fixed $d \geq 2$ and $r \geq 1$, we show that the height must grow extremely rapidly in terms of the girth.

2 Augmented trees

In this section and throughout, we restrict g to be even. If there is a (d, r, g)-graph, then let $m(d, r, g)$ denote the least height of the underlying tree in such a graph (otherwise, let $m(d, r, g) = \infty$). Theorem 1.3 is the statement that $m(d, r, g)$ is finite for all $d, r, g \in \mathbb{N}$. We prove this by double induction, using the following three lemmas.

Lemma 2.1. *For $d, r \in \mathbb{N}$, we have $m(d, r, 4) = 2r + 1$.*

Lemma 2.2. *For $g, d \in \mathbb{N}$ with g at least 4 and even, $m(d, 1, g + 2) \leq g + m(d, d^g, g)$.*

Lemma 2.3. *With d, r, g as above, $m(d, r+1, g) \leq m_1 + g - 1 + m_2$, where $m_1 = 2 \left\lfloor \frac{m(d,r,g)}{2} \right\rfloor + 1$ and $m_2 = m(d^{m_1+g}, 1, g)$.*

These three lemmas imply the finiteness of $m(d, r, g)$ for all $d, r, g \in \mathbb{N}$ with g even and at least 4. Letting $P(r, g)$ denote the claim that $m(d, r, g)$ is finite for all d, we prove $P(r, g)$ by induction on g. As the base step, $P(r, 4)$ holds for all r by Lemma 2.1. If $P(r, g)$ holds for all r, then we prove $P(r, g + 2)$ by induction on r: first $P(1, g + 2)$ holds by Lemma 2.2 (using the truth of $P(r, g)$ for all r), and then $P(r + 1, g + 2)$ follows from $P(r, g + 2)$ by Lemma 2.3 (since $P(1, g + 2)$ also holds). This completes the proof of Theorem 1.3.

It remains to prove the three lemmas. Lemma 2.1 is trivial: just make each leaf adjacent to its r non-parent ancestors at odd distance from it in the tree.

Proof of Lemma 2.2. Let G' with underlying tree T be a (d, d^g, g)-graph with height $m(d, d^g, g)$. Replace each leaf v of T with a complete d-ary tree T_v of height g rooted at v. Replace the augmenting edges from v to its ancestors by letting the d^g lower endpoints be the leaves of T_v instead of v. This produces a 1-augmented complete d-ary tree G of height $g + m(d, d^g, g)$. Since g is even, G is bipartite.

A cycle C in G must contain an augmenting edge. Contracting the added subtrees of height g to leaves of T contracts C to a closed walk C' in G'. Starting from a leaf of T, the walk C' must eventually revisit that leaf. Some portion C'' of C' revisits a leaf v of T with no other repeat visit to a leaf inside it; C'' is a cycle in G'. Hence the length of C'', whose edges appear in C, is at least g. Viewed as edges of G, the two edges of C'' incident to v in G' have different endpoints in T_v. In addition to the edges of C'', in C at least two edges in T_v must be added joining these endpoints. Hence C has length at least $g + 2$. $\qquad\square$

For Lemma 2.3, most of the work is in describing the construction; it will be easy to show that the construction has the desired properties.

Proof of Lemma 2.3. Fix r. Assuming for all d and g that $m(d, r, g)$ and $m(d, 1, g)$ are finite, let $m_1 = 2 \lfloor m(d, r, g)/2 \rfloor + 1$ and $m_2 = m(d^{m_1+g}, 1, g)$. Note that m_1 is the least odd integer that is at least $m(d, r, g)$. We construct the desired graph G from graphs G_1, G_2, and G_3.

Let $d' = d^{m_1+g}$. Begin with a $(d', 1, g)$-graph H having height m_2. Form an induced subgraph G_1 by starting from the root of the underlying tree of H and keeping only d children of each included vertex, except that all d' children are kept at the last level. Thus G_1 has an underlying tree T_1 of height m_2, and deleting the $d^{m_2-1}d'$ leaves of T_1 yields a complete d-ary tree of height $m_2 - 1$. Each leaf of T_1 has one ancestor as a neighbor in G_1.

Let G_2 be a (d, r, g)-graph having height m_1. If $m(d, r, g)$ is even, then $m_1 = m(d, r, g) + 1$. In that case, form G_2 from d copies of the given construction with height $m(d, r, g)$ by adding one vertex having their roots as children.

Let G_3 be the graph obtained from a complete d-ary tree of height g by adding at each leaf a copy of G_2 rooted at that leaf. Thus G_3 is a (d, r, g)-graph having height $g + m_1$ in which the augmenting edges lead only to vertices whose distance from the root is at least g.

Finally, let G be the graph obtained as follows. In G_1, let $S(u)$ be the star consisting of a vertex u at level $m_2 - 1$ and its d' leaf children. Expand each $S(u)$ into a copy $G_3(u)$ of the graph G_3, so that the d' leaves in G_3 each become one of the leaves in $S(u)$, inheriting the one augmenting edge incident to that leaf in G_1. We call these augmenting edges obtained from G_1 *long edges*; the augmenting edges in $G_3(u)$ are *short edges*.

The underlying tree for our construction has three parts. The top part is the tree for G_1 without its bottom level; it has height $m_2 - 1$. The middle part has height g and receives no augmenting edges from leaves; it consists of the top portions of copies of G_3. The bottom part, having height m_1, consists of copies of G_2 within copies of G_3. For each leaf, r short edges connect it to the bottom part, and one long edge connects it to the top part. Thus G is an r-augmented complete d-ary tree of height $m_1 + g + m_2 - 1$. When replacing the augmenting edge from a leaf of G_1 by a long edge, the difference in the heights of the endpoints increases by $m_1 + g - 1$. Since m_1 is odd, this change is even, so G is bipartite.

A cycle C in G that contains no long edges is a cycle in a copy of G_2 and hence has length at least g. When C contains a long edge, contracting the subtrees of the form $G_3(u)$ into the stars $S(u)$ contracts C to a closed walk C' in G_1 using an augmenting edge. Starting from a leaf of the underlying tree T_1 along such an edge, the walk C' must eventually revisit that leaf. Some portion C'' of C' revisits a leaf v of T with no other repeat visit to a leaf inside it; C'' is a cycle in G_1. Hence the length of C'' is at least g. The edges of C'' appear in C except possibly for a subpath within T_1 joining two leaves. Each such portion lifts back to a portion of C that is at least as long. Hence C has length at least g.

This completes the proof of Lemma 2.3 and Theorem 1.3. $\square$

3 Applications

In a complete k-ary tree, a *full path* is a path from the root to a leaf. Let $[k] = \{1, \ldots, k\}$. A $[k]$-*coloring* is a k-coloring using the colors in $[k]$.

Definition 3.1. Given an ordering of the children at each internal vertex, the vertices of a complete k-ary tree with height m correspond naturally to the strings of length at most m from the alphabet $[k]$. Define an edge-coloring ϕ by letting the color of each edge from parent x to child y be the index of y in the ordering of the children of x (note that ϕ is not a proper coloring). For a $[k]$-coloring f of T, a full path P is an f-*path* if the color of each non-leaf vertex on P equals the color of the edge to its child on P.

Whenever f is a $[k]$-coloring of a complete k-ary tree, there is a unique f-path: just start from the root and repeatedly follow the descending edge whose color matches the color of the current vertex. Similarly, every full path is an f-path for some $[k]$-coloring f.

3.1 Large chromatic number and girth

As mentioned in the introduction, there exist t-uniform hypergraphs with large chromatic number and girth. Our (d, r, g)-graphs provide a remarkably simple such construction. It has the benefits of being non-recursive (once (d, r, g)-graphs are constructed), and not involving hypergraphs as inputs to the construction. Thus unlike the earlier constructions which use hypergraphs to provide high girth graphs, the method described here constructs high girth graphs and hypergraphs using only graphs.

Theorem 3.2 ([3, 4, 12, 13, 11, 8]). *For $k, g, t \in \mathbb{N}$, there is a t-uniform hypergraph with girth at least g and chromatic number larger than k.*

Proof. Let G be a $(k, (t-1)k+1, 2g)$-graph with underlying tree T having leaf set L. Let $V' = V(T) - L$. For $v \in L$, consider the full path P ending at v. Among the $(t-1)k+1$ neighbors of v via augmenting edges, the pigeonhole principle yields a set of t neighbors of v whose descending edges along P have the same color; let e_v be such a set of vertices in V'. Let H be the t-uniform hypergraph with vertex set V' and edge set $\{e_v \colon v \in L\}$.

Any $[k]$-coloring f of V' yields a unique f-path in T, ending at some leaf v. As a coloring of H, this makes the edge e_v monochromatic. Hence H has no proper k-coloring.

Let C be a shortest cycle in H, with edges $e_1, \ldots, e_l$ in order and vertex x_i chosen from $e_{i-1} \cap e_i$ (subscripts modulo l). Since C is a shortest cycle, $x_1, \ldots, x_l$ are distinct. Each edge of H consists of neighbors of a single leaf of T via augmenting edges; let v_i be the common leaf neighbor of the vertices in e_i. Form C' in G by replacing each edge e_i of C by the copy of P_3 in G having endpoints x_{i-1} and x_i and midpoint v_i. Since for each leaf of T we formed exactly one edge in H, the leaves $v_1, \ldots, v_l$ are distinct. Hence C' is a cycle, and its length is twice that of C. By the choice of G as a $(d, k, 2g)$-graph, H has girth at least g. □

The hypergraph H in Theorem 3.2 satisfies $|E(H)| = |L| = k^h$ and $|V(H)| = |V'| = \frac{k^h - 1}{k - 1}$, where $h = m(k, (t-1)k+1, 2g)$. Hence $|E(H)| = (k-1)|V(H)| + 1$. However, H may have (and actually does have) dense subgraphs. For $t = 2$, we provide a different construction, inductive, of sparse graphs with large girth and chromatic number. A graph G is sparse when it has a small value of the *maximum average degree*, defined to be $\max_{H \subseteq G} \frac{\sum_{v \in V(H)} d_H(v)}{|V(H)|}$.

Definition 3.3. Let G be a (d, r, g)-graph with a specified ordering of the d children at each non-leaf vertex of the underlying tree T. The corresponding *reduced (d, r, g)-graph H* is obtained from G as follows: given the coloring ϕ of $E(G)$ from Definition 3.1, form H from G by deleting at each non-root internal vertex v of T the subtree under the descending edge whose color under ϕ is the same as the color of the edge to the parent of v. Each non-leaf vertex of $H \cap T$ has degree d in T, and ϕ is a proper edge-coloring of $H \cap T$.

The reduced (d, r, g)-graph with underlying tree T associated with the edge-coloring ϕ as in Definition 3.3 still has a unique f-path for any proper $[d]$-coloring f of T.

Theorem 3.4. *For $k, g \in \mathbb{N}$, there is a graph with girth at least g that is not k-colorable and has maximum average degree at most $2(k-1)$.*

Proof. For fixed g, we construct such a graph J_k by induction on k. For the basis step, let J_2 be an odd cycle of length at least g. Given J_{k-1}, let $r = |V(J_{k-1})|$.

Let H be a reduced $(k, (r-1)k+1, g)$-graph, with underlying tree T and edge-coloring ϕ. For each leaf v of T, consider the full path P ending at v. By the pigeonhole principle, some r neighbors of v in H (via augmenting edges) have the same color on their descending edges along P. Keep the augmenting edges from v to one such set and delete the other augmenting edges. The resulting graph H' is a reduced (k, r, g)-graph.

Next replace each leaf v of H' with a copy of J_{k-1}; each vertex in the copy for v inherits exactly one augmenting edge of H' from v. This is the graph J_k. The edge to v in T disappears; vertices at the level just before the leaves no longer have edges to children.

Any proper $[k]$-coloring f of $V(T)$ yields a unique f-path; it ends at some leaf v. Because it is an f-path, the colors on the vertices match the colors on the descending edges. Let Q be the copy of J_{k-1} corresponding to v in J_k. By the construction of J_k, there is a fixed color c that appears on the neighbor in $V(T)$ of each vertex in Q. Since J_{k-1} is not $(k-1)$-colorable, we cannot complete a proper k-coloring of J_k.

A cycle in one copy of J_{k-1} has length at least g. For any other cycle C in J_k, contracting each copy of J_{k-1} to a single vertex yields a closed walk C' in H' using some augmenting edge. Starting from a leaf of T along such an edge, the walk C' must eventually revisit that leaf. As usual, a portion of C' that does not include revisits to other leaves within it is a cycle in H' and therefore has length at least g; hence C has length at least g.

For the maximum average degree, consider a subgraph F, and let $F' = F - V(T)$. Being contained in copies of J_{k-1}, the graph F' has average degree at most $2(k-2)$. Augmenting edges add at most 1 to the degree of each vertex of F' and hence at most 2 to the degree-sum

in F for each vertex in F'. Working upward in T, each added vertex in F adds at most $k-1$ downward edges, which contributes at most $2(k-1)$ to the degree-sum. The root may add k downward edges, but the lowest vertex added from T adds fewer than $k-1$. Thus the degree-sum is at most $2(k-1)$ per vertex of F. $\qquad\square$

3.2 Choosability

A modification of the construction in Theorem 3.4 yields non-k-choosable bipartite graphs that are as sparse as can be. As noted in Theorem 1.1, every bipartite graph with maximum average degree at most $2(k-1)$ is k-choosable. Hence the graphs we construct in Theorem 1.2 with just one extra edge are $(k+1)$-choice-critical.

It is well known (since [5]) that a bipartite graph consisting of two even cycles sharing one vertex is not 2-choosable; indeed, it is 3-choice-critical.

Theorem 1.2. *For $k \geq 2$ and $g \geq 4$, there is a bipartite graph G_k with girth at least g that is not k-choosable even though every proper subgraph has average degree at most $2(k-1)$.*

Proof. We proceed by induction on k for even g. To count edges in subgraphs, we will orient G_k and count edges by their tails. The orientation gives each vertex outdegree $k-1$ except a designated root vertex, which has outdegree k, and every vertex will be reachable from the root. Thus G_k will have $(k-1)|V(G_k)|+1$ edges, and every proper subgraph will have smaller outdegree at some vertex and thus have average degree at most $2(k-1)$.

Let G_2 be the graph consisting of two g-cycles sharing one vertex, which is the root. Orient G_2 consistently along each of the two cycles. The desired properties hold.

For $k \geq 3$, suppose that G_{k-1} has all the desired properties. Let $r = |V(G_{k-1})| - 1$, and let H' be a reduced $(k,r,2g)$-graph, with underlying tree T. We modify the bipartite graph H' slightly to guarantee that G_k will be bipartite. Let (A,B) be the bipartition of G_{k-1}, with A containing the root, and let $a = |A| - 1$ and $b = |B|$. Each leaf v in T has $a+b$ incident augmenting edges. Let $A(v)$ denote some set of a of these edges. For the remaining b augmenting edges incident to v, move their endpoints in the tree one step closer to v along the full path to v. Let $B(v)$ denote this new set of b augmenting edges at v. Let H be the resulting graph; H is a reduced (k,r,g)-graph except for not being bipartite.

Form G_k from H by adding a copy of G_{k-1} for each leaf v of T, merging v with the root of G_{k-1}, with each vertex of A in the copy of G_{k-1} (other than the root) inheriting one edge of $A(v)$ and each vertex of B in the copy of G_{k-1} inheriting one edge of $B(v)$. Since the vertices of B have odd distance from v in G_{k-1}, this guarantees that G_k is bipartite.

Designate the root of T as the root of G_k. Orient the edges of T away from the root, keep the orientation guaranteed by the induction hypothesis on the copies of G_{k-1}, and orient the augmenting edges away from the copies of G_{k-1}. Because H' is a reduced $(k,r,2g)$-graph, every vertex has outdegree $k-1$ except that the root has outdegree k.

A cycle in a copy of G_{k-1} has length at least g. Let C be a cycle in G_k that is not in G_{k-1}. Contracting the copies of G_{k-1} in G to single points turns C into a closed walk in

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H. This walk contains a cycle C', since it uses some vertex of T, and vertices only have one edge into any copy of G_{k-1}. Thus C has length at least g, since H has girth at least g.

Let L' be an assignment of lists of size $k-1$ to G_{k-1} such that G_{k-1} is not L'-colorable and none of these lists intersects $[k]$. Form a list assignment L for G_k as follows. Put $L(x) = [k]$ for each non-leaf vertex x in $V(T)$. For each leaf $v \in V(T)$ and each vertex w of $V(G_{k-1})$, let w_v denote the copy of w in the copy of G_{k-1} at v. Let P be the full path in T ending at v. Let $L(w_v) = L'(w) \cup \{c\}$, where c is the color on the edge of P descending from the neighbor of w_v in $V(P)$. In particular, when w is the root, the added color is the color on the edge of T reaching v.

Let f be a coloring of G_k with $f(u) \in L(u)$ for $u \in V(G_k)$. If f is proper on T, then since $f(x) \in [k]$ for $x \in V(T)$, there is a unique f-path P in T. In the copy of G_{k-1} for the leaf v at the end of P, the color c that was added to each list is now forbidden in a proper coloring, leaving the list $L'(w)$ at w_v. By the choice of L', a proper coloring cannot be completed from these lists. □

3.3 Restricted list colorings

As described in the introduction, we now strengthen Theorem 1.2 by proving non-choosability results for restricted list assignments. We consider both restrictions on the intersections of adjacent lists and restrictions on the size of the union of the lists.

Every graph is L-colorable (by choosing arbitrarily) when adjacent vertices have disjoint lists, but L-colorability may fail when adjacent lists are almost disjoint. List coloring with intersection constraints on adjacent lists has been studied by Kratochvíl, Tuza, and Voigt [10] and by Füredi, Kostochka, and Kumbhat [6]. We next strengthen Theorem 1.2 by showing that our graph G_k there fails to be L-colorable for a particular k-list assignment L such that $|L(u) \cap L(v)| = 1$ for every edge uv.

Theorem 3.5. *Fix $g \in \mathbb{N}$ with $g \equiv 4 \pmod 6$. For $k \geq 2$, the bipartite graph G_k with girth at least g constructed in Theorem 1.2 admits a k-list assignment L such that G_k is not L-colorable despite satisfying $|L(u) \cap L(v)| = 1$ for all $uv \in E(G_k)$.*

Proof. For $k = 2$, let u be the common vertex of the two cycles in G_2. Set $L(u) = \{1, 2\}$. On each of the two cycles, the number of remaining vertices is a multiple of 3. Along one cycle, rotate through the lists $\{1, 3\}, \{3, 4\}, \{4, 1\}$. This forces color 1 onto a neighbor of u. On the other cycle substitute 2 for 1, forcing color 2 onto a neighbor of u. Now u cannot be colored. Adjacent lists share one color.

For $k \geq 3$, let T be the underlying tree in G_k. Color the edges of T by distinct colors. For a non-leaf vertex x in T, let $L(x)$ be the set of colors on the edges incident to x; thus lists adjacent via edges of T have one common color.

By the induction hypothesis, there is a $(k-1)$-list assignment L' on G_{k-1} such that G_{k-1} is not L'-colorable. For each leaf $v \in V(T)$, let L'_v be a copy of this assignment indexing the colors by v, so that the colors used for the copy G' of G_{k-1} at v will not be used anywhere

else. For each vertex w of $V(G_{k-1})$ other than the root, let w_v denote the copy of w in G'. Let P be the full path in T ending at v. Let x be the neighbor of w_v in $V(P)$, and let c_x be the color of the edge in P descending from x along P. Let $L(w_v) = L'_v(w) \cup \{c_x\}$. Let $L(v) = L'_v(v) \cup \{c_v\}$, where c_v is the color of the edge incident to v in T.

For any proper coloring f of T chosen from these lists, there is a unique full path Q such that the color of each non-leaf vertex is the color of the edge to its child on Q, constructed from the root: that is, an f-path. Let v be the leaf reached by Q. The parent of v has been given color c_v, so that color cannot be used at v. Similarly, for each other vertex in the copy of G_{k-1} at v, the added color in its list has been used on its neighbor in T. Finding an L-coloring of G_k thus requires finding an L'-coloring of G_{k-1}, which does not exist. $\qquad\square$

Perhaps surprisingly, for bipartite graphs larger intersections than in Theorem 3.5 also guarantee L-colorability, giving the sharpness of Theorem 3.5 in another way.

Proposition 3.6. *If G is a bipartite graph, and L is a list assignment such that any two adjacent lists have at least two common elements (the lists may have any sizes at least 2), then G is L-colorable.*

Proof. Let X and Y be the parts of G, and index the colors in $\bigcup_{v \in V(G)} L(v)$ as $c_1, \ldots, c_t$. Color each vertex of X with the highest-indexed color in its list and each vertex of Y with the lowest-indexed color in its list. If two adjacent vertices receive the same color, then it is the only common color in their lists, a contradiction. Hence the coloring is proper. $\qquad\square$

When G is j-colorable but not k-choosable, one may ask how large the union U of the lists must be in a k-list assignment L such that G is not L-colorable. Trivially $|U| > j$ is needed. In fact, one needs somewhat more, which reduces to $2k - 1$ when $j = 2$.

Proposition 3.7. *Let G be a j-colorable graph, with $j \le k$. If L is a k-list assignment on G such that $|\bigcup_{v \in V(G)} L(v)| \le \frac{j(k-1)}{j-1}$, then G is L-colorable. Furthermore, the bound is sharp.*

Proof. Let f be a proper j-coloring of G. Let $U = \bigcup_{v \in V(G)} L(v)$. Split U into disjoint sets $U_1, \ldots, U_j$, with the smallest having size $\lfloor |U|/j \rfloor$. Since $|U| \le \frac{j(k-1)}{j-1}$, the largest $j - 1$ of the sets together have size at most $k - 1$. (Note that $\left\lfloor \frac{j(k-1)}{j-1} \right\rfloor - \left\lfloor \frac{k-1}{j-1} \right\rfloor = k - 1$, and when $|U| < \left\lfloor \frac{j(k-1)}{j-1} \right\rfloor$ the conclusion becomes easier.) Thus each k-list $L(v)$ intersects each U_i. Hence each vertex v can choose a color from $L(v) \cap U_{f(v)}$. Such a coloring is proper.

For sharpness, consider a universe U of colors, and let G be a complete j-partite graph with $\binom{|U|}{k}$ vertices in each part. Assign lists by letting L give each k-subset of U as a list to one vertex in each part. In an L-coloring, each color can be chosen in only one part. Since a color must be chosen from every vertex, on each part at least $|U| - (k - 1)$ colors must be chosen. Hence $j(|U| - k + 1)$ colors must be chosen. Thus L-colorability requires $j(|U| - k + 1) \le |U|$, which is precisely the inequality $|U| \le \frac{j(k-1)}{j-1}$. $\qquad\square$

The sharpness examples in Proposition 3.7 are very dense and have small cycles. The special case $j = 2$ states that a bipartite graph is L-colorable when L is a k-list assignment with $|\bigcup_{v \in V(G)} L(v)| \le 2k - 2$. This condition forces any two lists to have at least two common elements, so Proposition 3.6 is stronger than Proposition 3.7 for the case $j = 2$. Nevertheless, we show next that Proposition 3.7 remains sharp when $j = 2$ even for sparse graphs with large girth having just one extra edge beyond where Theorem 1.1 applies.

Theorem 3.8. *Fix $k, g \in \mathbb{N}$ with g even and $k \ge 2$. There is a bipartite graph H_k and a k-list assignment L on H_k such that H_k is not L-colorable, even though $|\bigcup_{v \in V(H_k)} L(v)| = 2k - 1$ and H_k has girth at least g with each proper subgraph having average degree at most $2(k-1)$.*

Proof. We use induction on k. For $k = 2$, let H_2 be G_2, the graph consisting of two g-cycles sharing one vertex u. Set $L(u) = \{1, 2\}$. On one cycle, use lists $\{1, 3\}$ and $\{1, 2\}$ on the neighbors of u and $\{2, 3\}$ on the rest of the cycle. Since the number of copies of $\{2, 3\}$ is odd, color 1 must be chosen on a neighbor of u. Interchanging 1 and 2 yields the lists on the other cycle, forcing a neighbor of u to have color 2. Now u cannot be colored. The union of the lists has three colors.

For $k \ge 3$, let $r = |V(H_{k-1})| - 1$, and let $a + 1$ be the number of vertices of H_{k-1} in the partite set containing the root; note that $a < r$. We construct H_k with a list assignment L. Consider a reduced $(k, (r-1)k, 2g)$-graph with underlying tree T and corresponding proper $[k]$-edge-coloring of T. The root of T will be the root of H_k.

For each leaf v of T, proceed as follows. Let P be the full path to v in T. Since v has more than $(a - 1)k$ augmenting edges, by the pigeonhole principle there are a such edges for which the edge along P descending from the neighbor of v has the same color; call it c. Move the other endpoints of all $(r-1)k - a$ other augmenting edges at v one step closer to v along P, as in the proof of Theorem 1.2. Since $(r-1)k - a > (r - a - 1)k$, by the pigeonhole principle there are $r - a$ of these remaining edges for which the edge along P descending from the neighbor of v has the same color; call it c'. Discard all augmenting edges not chosen in these two steps. After doing this for each leaf v of T, the result is a reduced (k, r, g)-graph except for not being bipartite.

For each leaf v of T, add a copy H'_v of H_{k-1}, merging its root with v and letting each non-root vertex inherit one of the augmenting edges at v, with the vertices in the part opposite v inheriting the $r - a$ edges whose other endpoints were moved closer to v. Let H_k be the resulting graph; it is bipartite, and the density bound for its subgraphs is computed as for G_k in Theorem 1.2. Arguing as for G_k also shows that H_k has girth at least g.

Next we produce the list assignment L. Assign list $[k]$ to each non-leaf vertex of T. By the induction hypothesis, for each leaf v of T there is a $(k-1)$-list assignment L'_v on H' whose lists are contained in a $(2k - 3)$-set. For this $(2k - 3)$-set use $[2k - 1] - \{c, c'\}$, discarding any additional color if $c' = c$. Also, let c_v be the color of the edge reaching v in T. Since $2k - 3 > k - 1$ when $k > 2$, we may permute the colors within L'_v to ensure that L' does not assign color c_v to v.

To define lists, form $L(v)$ by adding c_v to the list given by L'_v to the root. For $w \in V(H_{k-1})$ other than the root, let w_v be the copy of w in H'_v. Set $L(w_v) = L'_v(w) \cup \{c\}$ if w is in the same partite set as the root of H_{k-1}, and otherwise set $L(w_v) = L'_v(w) \cup \{c'\}$.

It remains to show that H_k is not L-colorable. Let f be a proper coloring chosen from L. Since the list on each non-leaf vertex of T is $[k]$ and the coloring is proper, there is a unique f-path Q leading to a particular leaf v. Since the color of each non-leaf vertex on Q agrees with the color on the edge descending from it along Q, the color added to the list of each vertex w_v in the copy of H_{k-1} at v has been used on its neighbor in T and is now forbidden from use on w_v. Finding an L-coloring of H_k thus requires finding an L'-coloring of H_{k-1}, which does not exist. □

4 The height of the trees in Theorem 1.3

The underlying trees in our construction of (d, r, g)-graphs are astoundingly tall; their height in terms of g is a version of the Ackermann function. Here we show that even for $r = 1$ and $d = 2$, they must be very tall. In the discussion below all logarithms are in base 2.

Theorem 4.1. *If G is a $(2, 1, g)$-graph with height m, then $g \leq (4 + o(1)) \log(\log^* m)$.*

Proof. For simplicity, we omit floor and ceiling signs; they are not crucial.

For $g \in \mathbb{N}$, let $q = 2^{g/4-2}$. Let $k_{-1} = -1$, $k_0 = g - 1$, and for $0 \leq i < r$ set

$$k_{i+1} = 2^{(k_i - g/2 + 4)/2} + k_i.$$

This yields $g \approx 4 \log(\log^* k_q)$. Let G be a 1-augmented binary tree of height m, and let g be the least integer such that $k_q \geq m$. We will find in G a cycle of length at most g.

Define integer intervals $I_0, \ldots, I_q$ by $I_j = [m - k_j, m - k_{j-1} - 1]$ (deleting any negative elements). These intervals group the levels in T. The number of levels in I_j is at most $k_j - k_{j-1}$, the value of which is roughly a tower of height j. However, since we only choose g so that $k_q \geq m$, the least j with $k_j \geq m$ may be less than q, so the intervals toward the end of the list may be empty.

Let the *mate* of a leaf of T be the other endpoint of its augmenting edge in G. Let the *type* of the leaf be j if the level of its mate lies in I_j. We may assume that no leaf has type 0, since otherwise G has a cycle of length at most g. With each leaf having type in the integer interval $[1, q]$, some type is assigned to at least $1/q$ of the leaves of G. Fix such a type t.

By averaging, for some vertex u at level $m - k_{t-1} - 1$ at least $1/q$ of the leaves under u have type t. Let C denote the set of all leaves of type t under u. Let v be the ancestor of u at level $m - k_t$ (or level 0 if $m < k_t$). For each leaf $x \in C$, the mate of x is on the u, v-path P in T. Note that $|V(P)| \leq k_t - k_{t-1} = 2^{(k_{t-1} - g/2 + 4)/2}$.

The vertex u has $2^{k_{t-1} - g/4 + 2}$ descendants at level $m - (g/4 - 1)$; call this set D. The subtree rooted at any $y \in D$ has $2q$ leaves. Call y *full* if at least two leaves of T under y belong to C. Let $\beta|D|$ be the number of full vertices in D. The number of leaves under u

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is $2q|D|$. Allowing all leaves under full vertices of D and at most one leaf under non-full vertices, the number of leaves in C under u is at most $(2q\beta + 1)|D|$. The fraction of leaves under u in C is thus at most $\beta + \frac{1}{2q}$, but by the choice of u it is at least $1/q$. Thus $\beta \geq \frac{1}{2q}$.

Hence at least $2^{k_t-1-g/2+3}$ vertices of D are full. Under each full vertex of D some two leaves v and v' have mates in P. If v and v' have the same mate x, then x completes a cycle of length at most $2 + 2(g/4 - 1) < g$ with the path joining v and v' in T. Otherwise, each full vertex of D has two leaves under it whose mates are distinct vertices of P. Since the number of full vertices of D exceeds $\binom{|V(P)|}{2}$, by the pigeonhole principle some two vertices $y, y' \in D$ yield the same pair $x, x' \in V(P)$ of mates of two leaves under them. The paths joining those leaves in the subtrees under y and y' and the edges from those leaves to x and x' form a cycle of length at most $2(g/4 - 1) + 2(g/4 - 1) + 4$, which equals g. $\qquad\square$

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