A note on graph colorings and graph polynomials

Noga Alon [*] Michael Tarsi [†]

Abstract

It is known that the chromatic number of a graph $G = (V, E)$ with $V = \{1, 2, \ldots, n\}$ exceeds k iff the graph polynomial $f_G = \prod_{ij \in E, i<j}(x_i - x_j)$ lies in certain ideals. We describe a short proof of this result, using Ore's version of Hajós' theorem. We also show that a certain weighted sum over the proper k-colorings of a graph can be computed from its graph polynomial in a simple manner.

1 Introduction

The *graph polynomial* of a graph $G = (V, E)$ on a set $V = \{1, 2, \ldots, n\}$ of n vertices is $f_G = \prod_{ij \in E, i<j}(x_i - x_j)$. By definition, if the graph is not k-colorable and S is any set of k reals, this polynomial vanishes for all values of $(x_1, x_2, \ldots, x_n) \in S \times S \times \ldots \times S$, since for each such value of the variables x_i there is some edge $ij \in E$ with $x_i - x_j = 0$. This shows that the graph polynomial encodes some information about the chromatic number of the graph, and indeed it is shown in [4] and [1] that a graph is not k-colorable if and only if its graph polynomial lies in certain ideals. In the first half of this note we present a new, short proof of this result, stated in Theorem 1.1 and the two corollaries following it. The proof is based on Ore's version of the theorem of Hajós. It is worth mentioning that Ore noted in [5] (see also [3]) that this stronger version is sometimes crucial for applications, but gave no examples of such an application. Our proof here is such an example, as it seems much harder to deduce the theorem from the original version of the theorem of Hajós.

Theorem 1.1 *Let F be a field, and let I be an ideal of polynomials in $F[x_1, x_2, \ldots, x_n]$ which contains all the graph polynomials of complete graphs on any subset of cardinality $k+1$ of $\{1, 2, \ldots, n\}$. Then for any graph $G = (V, E)$ on the n vertices $\{1, 2, \ldots, n\}$ whose chromatic number exceeds k, the graph polynomial f_G lies in I.*

[*] AT & T Research, Murray Hill, NJ 07974, USA and Department of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel. Email: noga@math.tau.ac.il. Research supported in part by a grant from the Israel Science Foundation.

[†] Department of Computer Science, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel. Email: tarsi@math.tau.ac.il. Research supported in part by a grant from the Israel Science Foundation.

Proposed running head: Colorings and Graph Polynomials.

Two simple corollaries of the above theorem are the following known results.

Corollary 1.2 (Kleitman and Lovász, [4].) *A graph G is not k-colorable if and only if the graph polynomial f_G lies in the ideal generated by all graph polynomials of complete graphs on $k+1$ vertices.*

Corollary 1.3 (Alon and Tarsi, [1].) *A graph G on the n vertices $\{1, 2, \ldots, n\}$ is not k-colorable if and only if the graph polynomial f_G lies in the ideal generated by the polynomials $x_i^k - 1$, $(1 \leq i \leq n)$ (over the complex field, say).*

Let $G = (V, E)$ be a graph, $V = \{1, 2, \ldots, n\}$, let $f_G = \prod_{ij \in E, i<j}(x_i - x_j)$ be its graph polynomial, and let $\overline{f}_G$ be the remainder of this polynomial modulo the ideal generated by the polynomials $x_i^k - 1, 1 \leq i \leq n$.
Put $Z_k^n = \{0, 1, \ldots, k-1\}^n$. For a polynomial

$$P = P(x_1, \ldots, x_n) = \sum_{v \in Z_k^n} a_v \prod_{i=1}^{n} x_i^{v_i}.$$

Define $||P||_2^2 = \sum_{v \in Z_k^n} |a_v|^2$.

By the last corollary, $\chi(G) > k$ if and only if $\overline{f}_G$ is the zero polynomial. A stronger result is given in the following theorem, in which C is the set of all proper colorings c of G by the k colors $\{0, \ldots, k-1\}$.

Theorem 1.4 *In the above notation*

$$||\overline{f}_G||_2^2 = \frac{4^{|E|}}{k^n} \sum_{c \in C} \prod_{ij \in E, i<j} \sin^2[\frac{\pi(c(i) - c(j))}{k}]. \tag{1}$$

Note that the above formula provides a lower bound for the number of proper k-colorings of a graph as a function of $||\overline{f}_G||_2^2$. For the special case $k = 3$ this is an equality, showing that the precise number $|C|$ of proper 3-colorings satisfies

$$||\overline{f}_G||_2^2 = \frac{4^{|E|}}{3^n} |C| (3/4)^{|E|} = 3^{|E|-n} |C|.$$

2 Ore, Hajós and ideals of polynomials

Theorem 1.1 will be deduced from Ore's version [5] of Hajós' Theorem [2]. A graph is called *Ore-(k+1)-constructible* if it can be obtained from cliques of size $k+1$ using repeatedly the following operation. If G_1, G_2 are already obtained disjoint graphs with edges $x_1 y_1$ and $x_2 y_2$, respectively, and if $w_1, w_2, \ldots, w_t$ are distinct vertices of $G_1 - x_1$ and $z_1, z_2, \ldots, z_t$ are distinct vertices of $G_2 - x_2$, construct a new graph H by omitting the edges $x_1 y_1$ and $x_2 y_2$, identifying x_1, x_2, connecting y_1, y_2 by an edge, identifying z_i and w_i for all i, and replacing multiple edges, if any, by single ones. Ore showed

([5], see also [3]) that a graph is not k-colorable if and only if it contains an Ore $(k+1)$-constructible subgraph.

Proof of Theorem 1.1. By the above mentioned theorem, it suffices to show that if H is obtained from the graphs G_1, G_2 as described above, and the graph polynomials of G_1 and G_2 lie in I, so does the graph polynomial of H. To see this, let f_{G_1} and f_{G_2} denote the graph polynomials of G_1 and G_2, respectively, using the indices of the vertices after the identification. Let A denote the graph polynomial corresponding to all the common edges of the graphs G_1, G_2 after the identification. Let P denote the graph polynomial of the graph of all edges of $G_1 - x_1y_1$ besides these common ones, and let Q be the graph polynomial of the graph of all edges of $G_2 - x_2y_2$ besides these common ones. Clearly,

$$f_{G_1} = (x - y_1)PA, \qquad f_{G_2} = (x - y_2)QA \qquad f_H = (-1)^{\epsilon}(y_1 - y_2)PQA \quad \text{where} \quad (\epsilon \in \{0, 1\}).$$

Therefore, $f_H = (-1)^{\epsilon}(Pf_{G_2} - Qf_{G_1}) \in I$, completing the proof. $\Box$

Proof of Corollary 1.2. Let S be an arbitrary subset of size at most k of the field F. The graph polynomial of any complete graph on $k+1$ vertices obviously vanishes when each variable attains a value from S, as two of the variables get the same value. Thus, if the graph polynomial f_G lies in the ideal generated by the polynomials of complete graphs on $k+1$ vertices, then it vanishes whenever each variable attains a value of S and hence G is not k-colorable. Conversely, by Theorem 1.1, if G is not k-colorable then f_G lies in the desired ideal, as needed. $\Box$

Proof of Corollary 1.3. If f_G lies in the ideal generated by the polynomials $x_i^k - 1$ then it vanishes whenever each x_i attains a value which is a k^{th} root of unity. This means that in any coloring of the vertices of G by the k^{th} roots of unity, there is a pair of adjacent vertices that get the same color, implying that G is not k-colorable.

Conversely, suppose G is not k-colorable. In order to deduce from Theorem 1.1 that f_G lies in the ideal J generated by the polynomials $x_i^k - 1$, it suffices to prove that the graph polynomial of any complete graph on $k+1$ vertices lies in J. A typical graph polynomial of such a complete graph is a Vandermonde determinant $det\{(x_i^j) : 0 \le i, j \le k\}$. The last row of this matrix is the sum of the vector $\{(x_j^k - 1) : 0 \le j \le k\}$ with the all 1 vector, and this shows that the above determinant is a sum of two determinants one of which is 0, as it has two identical rows, and the other having the vector $\{(x_j^k - 1) : 0 \le j \le k\}$ as one of its rows. Expanding with respect to this row we conclude that the determinant lies in J, completing the proof. $\Box$

3 Counting proper colorings

Proof of Theorem 1.4. Suppose $k \ge 2$, Let $w = e^{2\pi i/k}$ be a primitive k-th root of unity and let

$$P = P(x_1, \ldots, x_n) = \sum_{v \in Z_k^n} a_v \prod_{j=1}^{n} x_j^{v_j}$$

be a polynomial. Recall that $||P||_2^2 = \sum_{v \in Z_k^n} |a_v|^2$. By Parseval's Formula

$$||P||_2^2 \cdot k^n = \sum_{v \in Z_k^n} |P(w^{v_1}, \ldots, w^{v_n})|^2. \tag{2}$$

Suppose, now, that $G = (V, E)$ is a graph with $V = \{1, \ldots, n\}$, let $f_G = \prod_{ij \in E, i<j}(x_i - x_j)$ be its graph polynomial and let $\overline{f}_G$ be the remainder of this polynomial modulo the ideal generated by the polynomials $x_i^k - 1, 1 \leq i \leq n$.

To derive (1) from (2) observe that for $v \in Z_k^n$, if the coloring $c(i) = v_i$ is not a proper coloring of G then

$$\overline{f}_G(w^{v_1}, \ldots, w^{v_n}) = f_G(w^{v_1}, \ldots, w^{v_n}) = 0,$$

whereas if it is a proper coloring then

$$|\overline{f}_G(w^{v_1}, \ldots, w^{v_n})|^2 = |f_G(w^{v_1}, \ldots, w^{v_n})|^2$$

$$= \prod_{ij \in E, i<j} [2 - 2\cos\frac{2\pi(c(i) - c(j))}{k}] = 4^{|E|} \prod_{ij \in E, i<j} \sin^2[\frac{\pi(c(i) - c(j))}{k}].$$

This completes the proof. □

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