Abstract

The graph removal lemma is an important structural result in graph theory. It has many different variants and is closely related to property testing and other areas. Our main aim in this paper is to develop removal lemmas of the same spirit for two dimensional matrices. A matrix removal lemma is a statement of the following type: fix a finite family $F$ of matrices over some alphabet $\Gamma$. Suppose that $M$ is an $n \times n$ matrix over $\Gamma$, such that for any two positive integers $s, t$ only $o(n^{s+t})$ of the $s \times t$ submatrices of $M$ are equal to matrices from $F$. Then one can modify no more than $o(n^2)$ entries in $M$ to make it $F$-free (that is, after the modification no submatrix of $M$ is equal to a matrix from $F$).

We prove matrix removal lemmas in several different scenarios. As a representative example, the following is one of our main results: fix an $s \times t$ binary matrix $A$. For any $\epsilon > 0$ there exists $\delta > 0$ such that for any $n \times n$ binary matrix $M$ that contains less than $\delta n^{s+t}$ copies of $A$ as submatrices, there exists a set of less than $c n^2$ entries of $M$ that intersects every $A$-copy in $M$. Moreover, this removal lemma is efficient: $\delta^{-1}$ is polynomial in $\epsilon^{-1}$.

The major difficulty in our case is that the rows and the columns of a matrix are ordered. These are the first removal lemma type results for two dimensional graph-like objects with a predetermined order. Our results have direct consequences in property testing of matrices: they imply that for several types of families $F$ and choices of the alphabet $\Gamma$, one can determine with good probability whether a given matrix $M$ is $F$-free or $\epsilon$-far from $F$-freeness (i.e. one needs to change at least an $\epsilon$-fraction of its entries to make it $F$-free) by sampling only a constant number of entries in $M$.

Our results generalize the work of Alon, Fischer and Newman [4] and make progress towards settling one of their open problems. The proofs combine a conditional regularity lemma for matrices proved in [4] with additional combinatorial and probabilistic ideas.

1 Introduction

1.1 Background and Main Results

The graph removal lemma (established by Rusza and Szemerédi [20]; see also [2, 3]) is a well known structural result on graphs that states the following: for any fixed graph $H$ on $h$ vertices and any $\epsilon > 0$ there exists $\delta > 0$ such that for any graph $G$ on $n$ vertices that contains less than $\delta n^h$ copies of $H$, one can destroy all $H$-copies in $G$ by removing less than $c n^2$ edges. In other words, there exists a set of less than $c n^2$ edges that intersects all copies of $H$ in $G$. The main tool used in the proof of the graph removal lemma is the celebrated Szemerédi graph regularity lemma [22].

The induced graph removal lemma, established in [3] by proving a stronger version of the graph regularity lemma, is a similar result for induced subgraphs: for any fixed graph $H$ on $h$
vertices and any $\epsilon > 0$ there exists $\delta > 0$ such that if the number of induced subgraphs on $h$ vertices in an $n$-vertex graph $G$ that are isomorphic to $H$ is less than $\delta n^h$ then one can add or remove less than $\epsilon n^3$ edges in $G$ to make it induced $H$-free. Here $G$ is said to be induced $H$-free if no induced subgraph of it is isomorphic to $H$.

There are many useful variants, quantitative strengthenings and extensions of the graph removal lemma. See [8] for an extensive summary of this topic. Our main goal in this paper is to derive results of the same spirit for matrices, where the row and column orders are important. These are the first removal lemma type results on matrices with a predetermined order. Note that removal lemmas for vectors (i.e. one dimensional matrices where the order is important) are generally easier to obtain; in particular, a removal lemma for vectors over a fixed finite alphabet can be derived from a removal lemma for regular languages proved in [5]. A removal lemma for partially ordered sets with a grid-like structure, which can be seen as a generalization of the removal lemma for vectors, can be deduced from a result of Fischer and Newman in [9], where they mention that this problem for submatrices is more complicated and not understood. Recently, Korman and Reichman [15] obtained a removal lemma for patterns in strings and images. A pattern must be taken from consecutive locations, whereas in our case the rows and columns of a submatrix need not be consecutive. The case of patterns behaves very differently than that of submatrices, and in particular, in the removal lemma for patterns the parameters are linearly related (for any alphabet size) unlike the case of submatrices (in which, for alphabets of 3 letters or more, the relation cannot be polynomial; see Theorem 1.5 below).

To state our results we will need some notation. An $m \times n$ matrix $M$ over $\Gamma$ is a function from $I \times J$ to $\Gamma$ where the index sets $I, J$ are totally ordered sets of sizes $m, n$ respectively. $M$ is said to be binary (ternary) if the alphabet is $\Gamma = \{0, 1\}$ ($\Gamma = \{0, 1, 2\}$ respectively). Suppose that the indices in $I$ are $a_1 < \ldots < a_m$ and the indices in $J$ are $b_1 < \ldots < b_n$, then the $(i,j)$ entry of $M$ is $M(a_i, b_j)$. Two matrices of the same dimensions are equal if all couples of corresponding entries are equal. For subsets $S \subseteq I, T \subseteq J$, the submatrix of $M$ on $S \times T$ is the function $M$ restricted to $S \times T$. The order on the submatrix is induced by the order on $M$. Note that the number of $s \times t$ submatrices of $M$ is $\binom{m}{s} \binom{n}{t} \leq m^s n^t$.

The density $t(A, M)$ of an $s \times t$ matrix $A$ in an $m \times n$ matrix $M$ is the fraction of $A$-copies among all $s \times t$ submatrices of $M$; if $s > m$ or $t > n$ we define $t(A, M) = 0$. The density of a finite family $F$ of matrices in $M$ is $t(F, M) = \sum_{A \in F} t(A, M)$. A matrix $M$ is $F$-free if $t(F, M) = 0$. The hitting number $h(F, M)$ of $F$ in $M$ is the minimal size of a set of entries in $M$ that intersects all copies of matrices from $F$ in $M$, divided by the total number of entries in $M$. Finally, the distance of $M$ from $F$-freeness, denoted by $d(F, M)$, is the minimal number of entries one needs to modify in $M$ to make it $F$-free, divided by the total number of entries in $M$. For a matrix $A$ we define $h(A, M) = h(\{A\}, M)$ and $d(A, M) = d(\{A\}, M)$.

A weak (strong) matrix removal lemma for a finite family of matrices $F$ over an alphabet $\Gamma$ is a statement with the following structure: for any $\epsilon > 0$ there exists $\delta > 0$ such that any square matrix $M$ over $\Gamma$ with $t(F, M) < \delta$ satisfies $h(F, M) < \epsilon$ ($d(F, M) < \epsilon$ respectively). If $\delta^{-1}$ is polynomial in $\epsilon^{-1}$ (where the constant in the exponent might depend on the family $F$ and the alphabet $\Gamma$), we say that the removal lemma is efficient. To simplify the discussion, we restrict our attention to the case where $M$ is square, but all of our results are generalizable to $m \times n$ matrices.

Our first main result is an efficient weak removal lemma for any finite family of binary matrices. It can be seen as an analogue of the (non induced) graph removal lemma for binary matrices.

**Theorem 1.1.** Fix a finite family $F$ of binary matrices. For every $\epsilon > 0$ there exists $\delta > 0$ such that any square binary matrix $M$ with $t(F, M) < \delta$ satisfies $h(F, M) < \epsilon$. Moreover, $\delta^{-1}$ is polynomial in $\epsilon^{-1}$.

permutations if for any \( A \in F \), any matrix created by permuting the rows (columns respectively) of \( A \) is in \( F \). \( F \) is closed under permutations if it is closed under row permutations and under column permutations.

**Theorem 1.2** [4]. Let \( F \) be a finite family of binary matrices that is closed under permutations. For any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that any square binary matrix \( M \) with \( t(F, M) < \delta \) satisfies \( d(F, M) < \epsilon \). Moreover, \( \delta^{-1} \) is polynomial in \( \epsilon^{-1} \).

The problem of determining whether there exists a strong removal lemma for any finite family \( F \) of binary matrices, even without the polynomial dependence, was raised in [4] and is still open.

**Problem 1.3.** Is it true that for any fixed finite family \( F \) of binary matrices the following holds: for any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that any square binary matrix \( M \) with \( t(F, M) < \delta \) satisfies \( d(F, M) < \epsilon \)? If the answer is positive, is it true that \( \delta^{-1} \) is polynomial in \( \epsilon^{-1} \)?

Theorem 1.1 implies that to settle Problem 1.3 it is enough to show the following. Fix a finite family \( F \) of binary matrices. Then for any \( \epsilon > 0 \) there exists \( \tau > 0 \) such that any square binary matrix \( M \) with \( h(F, M) < \tau \) also satisfies \( d(F, M) < \epsilon \). Moreover, if \( \tau^{-1} \) is polynomial in \( \epsilon^{-1} \) then in the statement of Problem 1.3, \( \delta^{-1} \) is polynomial in \( \epsilon^{-1} \).

Our second main result makes progress towards solving Problem 1.3 by generalizing the statement of Theorem 1.2 to any family \( F \) of binary matrices that is closed under row (or column) permutations.

**Theorem 1.4.** Let \( F \) be a finite family of binary matrices that is closed under row permutations (or under column permutations). For any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that any square binary matrix \( M \) with \( t(F, M) < \delta \) also satisfies \( d(F, M) < \epsilon \). Moreover, \( \delta^{-1} \) is polynomial in \( \epsilon^{-1} \).

Our proof of Theorem 1.4 is somewhat simpler than the original proof of Theorem 1.2. One of the main tools in the proof of Theorems 1.1 and 1.4 is an efficient conditional regularity lemma for matrices developed in [4] (see also [16]). In the proof of Theorem 1.4 we only use a simpler form of the lemma, which is also easier to prove. The statement of the lemma and the proofs of Theorems 1.1, 1.4 appear in Section 2.

The main consequence of Theorem 1.2 is an efficient induced removal lemma for bipartite graphs. Indeed, when representing a bipartite graph by its (bi-)adjacency matrix, a forbidden subgraph \( H \) is represented by the family \( F \) of all matrices that correspond to bipartite graphs isomorphic to \( H \). Note that \( F \) is indeed closed under permutations in this case. As opposed to the polynomial dependence in the above results, a construction of Behrend [6] can be used to show that in the graph removal lemma and the induced graph removal lemma, \( \delta^{-1} \) is super polynomial in \( \epsilon^{-1} \); see [1] for more details. The best known upper bound for the relation is of tower type ([7, 11]). Fischer and Rozenberg [10] observed that a similar super polynomial lower bound holds for ternary matrices (and for binary three dimensional matrices). Their proof, also based on a Behrend-type construction, gives the following.

**Theorem 1.5** [10]. There exist arbitrarily small values of \( \epsilon > 0 \) for which there exist a finite family \( F \) of ternary matrices that is closed under permutations and an arbitrarily large square ternary matrix \( M \) with \( h(F, M) \geq \epsilon \) and \( t(F, M) < (c/\epsilon)^{-\frac{1}{\log c/\epsilon}} \) where \( c > 0 \) is an absolute constant.

Theorem 1.5 implies that an efficient weak removal lemma for ternary matrices cannot be obtained, even when \( F \) is closed under permutations, as opposed to the binary case. In Subsection 3.1 we describe another construction that establishes Theorem 1.5, which is simpler than the original construction in [10].

The case of non-binary matrices is less understood. The results that we will present here hold over any alphabet \( \Gamma \). A weak removal lemma for families that are closed under permutations follows from the graph removal lemma using a suitable construction.
Proposition 1.6. Let $\Gamma$ be an arbitrary alphabet and let $F$ be a finite family of matrices over $\Gamma$ that is closed under permutations. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that any square matrix $M$ with $t(F, M) < \delta$ also satisfies $h(F, M) < \epsilon$.

Note that Theorem 1.5 implies that the dependence of $\delta^{-1}$ on $\epsilon^{-1}$ in Proposition 1.6 cannot be polynomial. The question whether the statement of Proposition 1.6 holds for any finite family $F$ is open. Here we state the question in the following equivalent but simpler form.

Problem 1.7. Is it true that for any alphabet $\Gamma$ and matrix $A$ over $\Gamma$ and any $\epsilon > 0$ there exists $\delta > 0$ such that any square matrix $M$ over $\Gamma$ with $t(A, M) < \delta$ satisfies $h(A, M) < \epsilon$?

Note that Theorem 1.1 settles the binary case with $\delta^{-1}$ polynomial in $\epsilon^{-1}$, whereas in the general case the dependence cannot be polynomial.

Our main theorem in this domain shows that Problem 1.7 is equivalent to another statement that looks more accessible. We need some definitions to describe it. Let $A, M$ be matrices of dimension $s \times t, m \times n$ respectively. Let $S$ be the submatrix of $M$ on $\{r_1, \ldots, r_s\} \times \{c_1, \ldots, c_t\}$ where $r_1 < \ldots < r_s$ and $c_1 < \ldots < c_t$. The $i$-height of $S$ for $i = 1, \ldots, s-1$ is $(r_{i+1} - r_i)/m$; the $i$-width of $S$ for $i = 1, \ldots, t-1$ is $(c_{i+1} - c_i)/n$.

Theorem 1.8. The following statements are equivalent.

1. For any alphabet $\Gamma$, matrix $A$ over $\Gamma$ and $\epsilon > 0$ there exists $\delta > 0$ such that any square matrix $M$ with $t(A, M) < \delta$ satisfies $h(A, M) < \epsilon$.

2. For any alphabet $\Gamma$, $s \times t$ matrix $A$ over $\Gamma$ and $\epsilon > 0$ there exists $\delta > 0$ such that for any square matrix $M$, if either all copies of $A$ in $M$ have $i$-height less than $\delta$ for some $1 \leq i \leq s-1$ or all copies of $A$ in $M$ have $j$-width less than $\delta$ for some $1 \leq j \leq t-1$, then $M$ satisfies $h(A, M) < \epsilon$.

The proofs of Proposition 1.6 and Theorem 1.8 appear in the first part of Section 3. Our next result holds over any alphabet and serves as an important tool in the proof of Theorem 1.4. A matrix is unfoldable if no two neighbouring rows in it are equal and no two neighbouring columns in it are equal. The folding of a matrix $A$ is the unique matrix $\hat{A}$ generated from $A$ by deleting any row of $A$ that is equal to its predecessor (i.e. to the row that comes before it in the ordering of the row indices) and then deleting any column of the resulting matrix that is equal to its predecessor. Note that $\hat{A}$ is unfoldable.

Lemma 1.9. Fix a matrix $A$. For any $\epsilon > 0$ there exist $n^0, \delta > 0$ such that for any $n \geq n^0$, any $n \times n$ matrix $M$ with $t(\hat{A}, M) \geq \epsilon$ also satisfies $t(A, M) \geq \delta$. Moreover, $n^0$ and $\delta^{-1}$ are polynomial in $\epsilon^{-1}$.

Lemma 1.9 implies that generally, to prove removal lemma type results it is enough to consider only families of unfoldable matrices. The proof appears in Section 4.

Remark. All of the above results are stated for two dimensional matrices, but Proposition 1.6, Theorem 1.8 and Lemma 1.9 can also be generalized to matrices in higher dimensions in a straightforward manner.

1.2 Applications: Ordered Bipartite Graphs and Property Testing

An ordered bipartite graph $G$ is defined as a bipartite graph between two totally ordered sets $U$ and $V$. A semi ordered bipartite graph is a bipartite graph between an ordered set $U$ and another set $V$ (not necessarily ordered). Ordered and semi ordered bipartite graphs can be represented, respectively, by a matrix and a matrix where the row order is not important. Hence we will use the same notation that was given for matrices. To simplify the discussion we only consider the case $|U| = |V| = n$. 

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Theorem 1.4 is essentially an efficient induced removal lemma for semi ordered bipartite graphs: it implies that for any finite family $F$ of semi ordered bipartite graphs, for any $\epsilon > 0$ there exists $\delta > 0$ (with $\delta^{-1}$ polynomial in $\epsilon^{-1}$) such that if $t(F,G) < \delta$ then $d(F,G) < \epsilon$, that is, $G$ can be made induced $F$-free by adding or removing less than $\epsilon n^2$ edges in it. In particular, any bipartite graph $H$ with parts $S,T$ such that $S$ is partially ordered and $T$ is unordered can be represented by such a family $F$ of semi ordered bipartite graphs, and so (using the above terminology) if $t(H,G) < \delta$ then $d(H,G) < \epsilon$, that is, $G$ can be made induced $H$-free by adding or removing less than $\epsilon n^2$ edges in it. The efficient induced removal lemma for bipartite graphs from [4] is a special case of this statement, as we can take the partial order to simply be the trivial order.

Solving Problem 1.3 would give an ordered bipartite graph induced removal lemma, and in particular, an induced removal lemma for any fixed bipartite graph with any couple of partial orders on its two parts.

Theorem 1.1 implies that given a finite family $F$ of ordered bipartite graphs, for any $\epsilon > 0$ there exists $\delta > 0$ (with $\delta^{-1}$ polynomial in $\epsilon^{-1}$) such that any ordered bipartite graph $G$ with both parts of size $n$ that satisfies $t(F,G) \leq \delta$ also satisfies $h(F,G) \leq \epsilon$, that is, there exists a set of $\epsilon n^2$ edges in $G$ that intersects all copies of ordered bipartite graphs from $F$ in $G$.

Our results have direct consequences in the field of Property Testing. This active field of study in computer science, initiated by Rubinfeld and Sudan [19] (see [13] for the graph case), is dedicated to finding fast algorithms to distinguish between objects that satisfy a certain property and objects that are far from satisfying this property. These algorithms are called testers. In particular, the general problem of testing graph properties was investigated extensively; see e.g. [12] for a good survey of this topic.

A property $P$ of matrices over alphabet $\Gamma$ is simply a collection of matrices over $\Gamma$. A matrix is $\epsilon$-far from $P$ if one needs to change at least an $\epsilon$-fraction of its entries to get a matrix that satisfies $P$. A two-sided ($\epsilon,t$)-tester for $P$ is a (probabilistic) algorithm that does the following: given an input square matrix $M$ of any size, it samples $t$ entries of $M$ and then distinguishes, with error probability at most $1/3$, between the case where $M$ satisfies $P$ and the case where $M$ is $\epsilon$-far from satisfying $P$. For matrices that are neither $\epsilon$-far from satisfying $P$ nor satisfy it, both answers are accepted. Note that such testers take a number of samples that is independent of the size of the input. A one-sided ($\epsilon,t$)-tester is defined similarly, but here if $M$ satisfies $P$ then the tester always accepts, and if $M$ is $\epsilon$-far from it then the tester rejects it with probability at least $2/3$. A property is ($\epsilon,t$)-testable if it has an ($\epsilon,t$)-tester.

In our case, the property is $F$-freeness: the collection of all $F$-free matrices. Theorem 1.4 implies that for any family $F$ of binary matrices that is closed under row permutations (or under column permutations), the property of $F$-freeness is ($\epsilon,t$)-testable by a simple one sided tester for any $\epsilon > 0$ where $t$ is polynomial in $\epsilon^{-1}$. The tester works as follows: let $k = k(F)$ be the largest row or column dimension of matrices from $F$. We iterate the following procedure $2\delta^{-1}$ times (where $\delta$ is as in the statement of Theorem 1.4): choose a $k \times k$ submatrix $S$ of $M$ at random, and check if $S$ is $F$-free. We answer that $M$ is $F$-free if and only if all iterations sampled an $F$-free submatrix. Clearly, if $M$ is $F$-free then the tester will accept as desired. If $M$ is $\epsilon$-far from $F$-freeness then the tester will reject with probability at least $1 - \epsilon^{-2} > 2/3$, as needed.

1.3 Additional Notation

Here we give some more notation that will be useful throughout the rest of the paper. We give the notation for rows but the notation for columns is equivalent. Let $M$ be an $m \times n$ matrix with index sets $I,J$; we sometimes simply assume that $I,J$ are the sets $[m],[n]$ with the standard $<$ order on integers. For two rows in $M$ whose indices in $I$ are $r < r'$, we say that row $r$ is smaller than row $r'$ and row $r'$ is larger than row $r$. The predecessor of row $r$ in $M$ is the largest row $\bar{r}$ in $M$ smaller than $r$. In this case we say that $r$ is the successor of $\bar{r}$.
2 Proofs for the Binary Case

This section is dedicated to the proof of our main results in the binary domain: Theorem 1.1 and Theorem 1.4. As a general remark for the proofs in this section, We may and will assume that a square matrix $M$ is sufficiently large (given $\epsilon > 0$), by which we mean that $M$ is an $n \times n$ matrix with $n \geq n_0$ for a suitable $n_0 > 0$ that is polynomial in $\epsilon^{-1}$.

One of the main tools in the proofs of this section is a conditional regularity lemma for matrices due to Alon, Fischer and Newman [4]. We will describe a simpler version of the lemma (this is Lemma 2.1 below) along with another useful result from their paper (Lemma 2.2 below). Combining these results together yields the original version of the conditional regularity lemma used in the original proof of Theorem 1.2 in [4]. It is worth to note that even though Theorem 1.4 generalizes Theorem 1.2, for its proof we only need the simpler Lemma 2.1 and not the original regularity lemma, whose proof requires significantly more work. Lemma 2.2 will only be used in the proof of Theorem 1.1.

We start with some definitions. A $(\delta,r)$-row-clustering of an $n \times n$ matrix $M$ is a partition of the set of rows of $M$ into $r + 1$ clusters $R_0, \ldots, R_r$ such that the error cluster $R_0$ satisfies $|R_0| \leq \delta n$ and for any $i = 1, \ldots, r$, every two rows in $R_i$ differ in at most $\delta n$ entries. That is, for every $e, e' \in R_i$, one can make row $e$ equal to $e'$ by modifying at most $\delta n$ entries. A $(\delta,r)$-column-clustering is defined analogously on the set of columns of $M$. The first conditional regularity lemma states the following.

**Lemma 2.1** [4]. Let $k$ be a fixed positive integer and let $\delta > 0$ be a small real. For every $n \times n$ binary matrix $M$ with $n > (k/\delta)^{O(k)}$, either $M$ admits $(\delta,r)$-clusterings for both the rows and the columns with $r \leq (k/\delta)^{O(k)}$, or for every $k \times k$ binary matrix $A$, at least a $(\delta/k)^{O(k^2)}$ fraction of the $k \times k$ submatrices of $M$ are copies of $A$.

Let $R$ be a set of rows and let $C$ be a set of columns in an $n \times n$ matrix $M$. The block $R \times C$ is the submatrix of $M$ on $R \times C$. A block $B$ is $\delta$-homogenous with value $b$ if there exists $b \in \{0,1\}$ such that at least a $1 - \delta$ fraction of the entries of $B$ are equal to $b$. A $(\delta,r)$-partition of $M$ is a couple $(R,C)$ where $R = \{R_1, \ldots, R_r\}$ is a partition of the set of rows and $C = \{C_1, \ldots, C_r\}$ is a partition of the set of columns of $M$, such that all but a $\delta$-fraction of the entries of $M$ lie in blocks $R_i \times C_j$ that are $\delta$-homogenous. The second result that we will need from [4], relating clusterings and partitions of a matrix, is as follows.

**Lemma 2.2** [4]. Let $\delta > 0$. If a square binary matrix $M$ has $(\delta^2/16, r)$-clusterings $R,C$ of the rows and the columns respectively then $(R,C)$ is a $(\delta, r + 1)$-partition of $M$.

For the proofs of the above lemmas see [4]. We continue to the proof of Theorem 1.1. The following lemma is a crucial part of the proof.

**Lemma 2.3.** Fix an $s \times t$ matrix $A$. For any $\epsilon > 0$ there exists $\tau > 0$ such that any $n \times n$ matrix $M$ with $h(A,M) \geq \epsilon$ either satisfies $t(B,M) \geq \tau$ for any $s \times t$ matrix $B$ or there exist subsets of indices $X,Y$ of sizes $s - 1, t - 1$ respectively such that $M$ contains $\tau n^2$ pairwise disjoint copies of $A$ that are separated by $X \times Y$. Moreover, $\tau^{-1}$ polynomial in $\epsilon^{-1}$.
Proof. The idea of the proof is to gradually find row separators and then column separators while maintaining a large set of pairwise disjoint copies of $A$ that conform to these separators. Let $\epsilon > 0$ and let $M$ be a large enough $n \times n$ binary matrix with $h(A, M) \geq \epsilon$, then $M$ contains a collection $U_0$ of $\epsilon n^2 / st$ pairwise disjoint $A$-copies. To see this, take $U_0$ to be a maximal such collection. The set of entries of $A$-copies in $U_0$ intersect any $A$-copy in $M$, giving that $|U_0| st \geq \epsilon n^2$.

We will prove the following claim by induction on $i$, for $i = 0, 1, \ldots, s - 1$: there exist $\tau_i, \delta_i$ with $\tau_i^{-1}, \delta_i^{-1}$ polynomial in $\epsilon^{-1}$ such that either $t(B, M) \geq \tau_i$ for any $s \times t$ matrix $B$ or there exist $0 = x_0 < x_1 < \ldots < x_i$ and a set $U_i$ of $\delta_{i}n^2$ pairwise disjoint $A$-copies in $M$ whose $j$-th row is bigger then $x_{j-1}$ and no bigger than $x_j$ for any $1 \leq j \leq i$, and the $(i + 1)$-th row is bigger than $x_i$. The base case $i = 0$ is trivial with $\delta_0 = \epsilon / st$. Suppose now that $i \geq 1$ and that $x_0, \ldots, x_{i-1}, \delta_{i-1}$ and $U_{i-1}$ are already determined. Applying lemma 2.1 on $M$ with parameters $k = \max\{s, t\}$ and $\delta_i^{-1}$, either $t(B, M) \geq \tau_i$ for any $s \times t$ matrix $B$ with $\tau_i^{-1}$ polynomial in $\epsilon^{-1}$ and we are done or $M$ has a $(\delta_{i-1}/4, r_i)$-row-clustering $R_i$ of $M$ for $r_i$ polynomial in $\delta_{i-1}$ and so in $\epsilon^{-1}$. The number of rows of $M$ that contain the $i$-th row of at least $\delta_i^{-1}n/2$ of the $A$-copies in $U_{i-1}$ is at least $\delta_i^{-1}n/2$, since the number of $A$-copies in $U_{i-1}$ whose $i$-th row is not taken from such a row of $M$ is less that $n \cdot \delta_i^{-1}n/2 = \delta_i^{-1}n^2/2$. Let $R_i$ be a row cluster that contains at least $\delta_i^{-1}n/2r_i$ such rows. Note that all of these rows are bigger than $x_{i-1}$.

Take subclusters $R^1_i, R^2_i$ of $R_i$, each containing at least $|\delta_i^{-1}n/4r_i| \geq \delta_i^{-1}n/5r_i$ such rows (the inequality holds for $n$ large enough) where each row in $R^1_i$ is smaller than each row in $R^2_i$. Take $x_i$ to be the row index of the biggest row in $R^1_i$.

Take arbitrarily $\delta_i^{-1}n/5r_i$ couples of rows $(r, r')$ where $r \in R^1_i$ and $r' \in R^2_i$ and every row participates in at most one couple. Let $(r, r')$ be such a couple. There exist $\delta_i^{-1}n/2$ $s \times t$ submatrices of $M$ that are $A$-copies from $U_{i-1}$ and whose $i$-th row is $r$. Moreover, for any $j < i$ the $j$-th row of each of these submatrices lies between $x_{j-1}$ (non-inclusive) and $x_j$ (inclusive). Since $r$ and $r'$ differ in at most $\delta_i^{-1}n/4$ entries, there are at least $\delta_i^{-1}n/4$ such submatrices $T$ that satisfy the following: if we modify $T$ by taking its $i$-th row to be $r'$ instead of $r$, $T$ will remain an $A$-copy. Moreover, after the modification, the $i$-th row of $T$ is in $R^1_i$ and is therefore no bigger than $x_i$, whereas the $(i + 1)$-th row of $T$ is bigger than the $i$-th row of $T$ before the modification which is bigger than $x_i$, as needed. For every couple $(r, r')$ we can produce $\delta_i^{-1}n/4$ pairwise disjoint copies of $A$ whose $j$-th row is between $x_{j-1}$ and $x_j$ for any $j \geq i$ and the $(i + 1)$-th row is after $x_i$. There are $\delta_i^{-1}n/5r_i$ such couples $(r, r')$, and in total we get a set $U_i$ of $\delta_i n^2$ copies of $A$ with the desired structure for $\delta_i = \delta_i^{-2} = 20r_i$ where $\delta_i^{-1}$ is polynomial in $\delta_i^{-1}$ and so in $\epsilon^{-1}$. Note that the copies in $U_i$ are pairwise disjoint. In the end of the process there is a set $U = U_s$ of $\delta_n n^2$ pairwise disjoint copies of $A$ whose rows are separated by $X = \{x_1, \ldots, x_s\}$. A feature that is useful for what follows is that each copy in $U$ has exactly the same set of columns (as a submatrix of $M$) as one of the original copies of $U_0$.

Now we apply the same process as above but in columns instead of rows, starting with the $\delta_s n^2$ copies in $U$. In the end of the process, we obtain that for some $\tau_i, \delta_i$ such that $\tau_i^{-1}$ and $\delta_i^{-1}$ are polynomial in $\delta_s^{-1}$ and so in $\epsilon^{-1}$, either $t(B, M) \geq \tau_i$ for any $s \times t$ matrix $B$ or there exists a set $U$ of $\delta_s n^2$ pairwise disjoint copies of $A$ whose columns are separated by a set of indices $Y$ of size $t - 1$. Moreover, by the above feature, each of the copies in $U$ has the same set of rows as some copy of $A$ from $U$, so each copy has its rows separated by $X$. Hence $X \times Y$ separates all copies in $U$. Taking $\tau = \min\{\tau_i, \delta_i\}$ finishes the proof. \qed

Next we show how Theorem 1.1 follows from Lemma 2.3.

Proof of Theorem 1.1. Let $F$ be a fixed family of binary matrices and let $k$ be the largest row or column dimension of a matrix from $F$. Let $\epsilon > 0$ and let $M$ be an $n \times n$ binary matrix with index sets $[n], [n]$ that satisfies $h(F, M) \geq \epsilon$ where $n > (k/\epsilon)^{O(k)}$. Observe that $h(F, M) \leq \sum_{A \in F} h(A, M)$ and so there exists an $s \times t$ matrix $A \in F$ for which $h(A, M) \geq \epsilon / |F|$. Lemma 2.3 implies that either $t(A, M) \geq \tau$ where $\tau^{-1}$ is polynomial in $\epsilon^{-1}$ (in this case we are
where \( \delta > 1.9 \) (that will be proved in Section 4), there exist 

\[ n \]

in it are unfoldable. The

Proof of Theorem 1.4.

It will be enough to prove the statement of the theorem only for 

\[ k \]

family of binary matrices that is closed under row permutations. Let 

\[ F \]

be the maximum row

or column dimension of a matrix from \( F \). Let \( \epsilon > 0 \) and apply Lemma 2.1 with parameters \( k \) and \( \epsilon/6 \). Let \( M \) be a large enough \( n \times n \) matrix with \( n > (k/\epsilon)^{O(k)} \), then either

\[ t(B, M) \geq \delta_2 \]

done) or \( M \) contains at least \( \tau n^2 \) pairwise disjoint copies of \( A \) separated by \( X \times Y \) for suitable index subsets \( X, Y \). By Lemma 2.1 we get that either \( M \) has \( (\tau^2/128, r) \)-clusterings of the rows

and the columns where \( r \) is polynomial in \( \tau^{-1} \) and so in \( \epsilon^{-1} \), or at least a \( \alpha = (\tau^2/128k)^{O(k^2)} \) fraction of the \( s \times t \) submatrices are \( A \); in the second case we are done. Suppose then that \( M \) has \( (\tau^2/128, r) \)-clusterings \( R, C \) of the rows, columns respectively. The next step is to create refinements of the clusterings. Write the elements of \( X, Y \) as

\[ \frac{r}{s} \]

fraction of the

\[ n \]

and the columns where

\[ x \]

separators

\[ A \]

of any

\[ \tau/n \]

couple

\[ A \]

4-homogenous blocks and agree with the value of the block in which they lie. Hence there exist sets of rows \( R_1, \ldots, R_s \in R \) and sets of columns \( C_1, \ldots, C_t \in C \) and a collection \( A \) of \( \tau n^2/2 (r')^{2k} \) pairwise disjoint \( A \)-copies separated by \( X \times Y \) such that for any \( 1 \leq i \leq s, 1 \leq j \leq t \), the block \( R_i \times C_j \) is \( \tau/4 \)-homogenous, has value \( A(i, j) \), lies between row separators \( x_{i-1} \) and \( x_i \) and between column separators \( y_{j-1} \) and \( y_j \), and contains the \((i, j)\) entry of any \( A \)-copy in \( A \). This implies that \( |R_i|, |C_j| \leq \tau n^2/2 (r')^{2k} \) for any \( 1 \leq i \leq s \) and \( 1 \leq j \leq t \), So there are \( (\tau/2(r')^{2k}) s t n^{s+t} \) \( s \times t \) submatrices of \( M \) whose \((i, j)\) entry lies in \( R_i \times C_j \) for any \( i, j \). Picking such a submatrix \( S \) at random, the probability that \( S(i, j) \neq A(i, j) \) for a specific couple \( i, j \) is at most \( \tau/4 \); thus \( S \) is equal to \( A \) with probability at least \( 1 - \sigma \tau/4 > 1/2 \) for small enough \( \tau \). Hence we get that

\[ t(F, M) \geq t(A, M) \geq (\tau/2(r')^{2k}) s t/2, \]

finishing the proof. \( \square \)

Next we give the proof of Theorem 1.4. For the proof, recall the definition of an unfoldable matrix and a folding of a matrix from Section 1. A family of matrices is unfoldable if all matrices in it are unfoldable. The folding of a finite family \( F \) of matrices is the set \( \tilde{F} = \{ \tilde{A} : A \in F \} \) of the foldings of the matrices in \( F \). Observe that \( \tilde{F} \) is unfoldable for any family \( F \). Note that if \( F \) is closed under row permutations then \( \tilde{F} \) is also closed under (row) permutations.

**Proof of Theorem 1.4.** It will be enough to prove the statement of the theorem only for unfoldable families that are closed under row permutations. Indeed, suppose that Theorem 1.4

is not \( F \)-free then it is not \( \tilde{F} \)-free as well, so we get that

\[ t(\tilde{F}, M) \geq \delta(c) \]

Therefore, there exists some \( A \in F \) such that \( t(A, M) \geq \delta(\epsilon) \). By Lemma 1.9 (that will be proved in Section 4), there exist \( n^0, \delta_1 \) such that if \( n \geq n^0 \) then \( t(A, M) \geq \delta_1 \), where \( \delta_1 \) and \( n^0 \) are polynomial in \( \delta^{-1} \) and thus in \( \epsilon^{-1} \). Therefore, if \( n \geq n_0 \) then we have

\[ t(F, M) \geq t(A, M) \geq \delta_1. \]

We can take small enough \( 0 < \delta \leq \delta_1 \) such that \( t(F, M) < \delta \) implies that

\[ t(F, M) = 0 \]

and so trivially \( d(F, M) = 0 < \epsilon \) for \( n \geq n^0 \), with \( \delta^{-1} \) polynomial in \( \epsilon^{-1} \) as needed.

Therefore, it is enough to consider only unfoldable families. Let \( F \) be an unfoldable finite family of binary matrices that is closed under row permutations. Let \( k \) be the maximum row or column dimension of a matrix from \( F \). Let \( \epsilon > 0 \) and apply Lemma 2.1 with parameters \( k \) and \( \epsilon/6 \). Let \( M \) be a large enough \( n \times n \) matrix with \( n > (k/\epsilon)^{O(k)} \), then either

\[ t(B, M) \geq \delta_2 \]


for any $k \times k$ matrix $B$ where $\delta_2^{-1}$ is polynomial in $\epsilon^{-1}$, implying that $t(F, M) \geq \delta_2$ as well, or $M$ has an $(\epsilon/6, r)$-clustering of the rows with $r$ polynomial in $\epsilon^{-1}$. In the first case we are done (by taking $\delta \leq \delta_2$ in the statement of the lemma), so suppose that $M$ has an $(\epsilon/6, r)$-clustering $R = \{R_0, \ldots, R_t\}$ of the rows where $R_0$ is the error cluster. We will show that in this case there exists $\delta \leq \delta_2$ (which will be determined later) such that if $t(F, M) < \delta$ then $d(F, M) < \epsilon$, where $\delta^{-1}$ is polynomial in $\epsilon^{-1}$, finishing the proof.

Suppose that $d(F, M) \geq \epsilon$. We say that a cluster $R \neq R_0$ in $R$ is large if it contains at least $en/6r$ rows. Note that the total number of entries that do not lie in large clusters is at most $en/6 + en/6 = en/3$. Pick an arbitrary row $r(R)$ from every large cluster $R \in R$ and denote by $Q$ the submatrix of $M$ created by these rows. Let $A(Q)$ be a collection of pairwise disjoint copies of matrices from $F$ in $Q$ that has the maximal possible number of copies. Suppose to the contrary that $|A| \leq en/3k$ and let $C$ be the set of all columns of $M$ that intersect a copy from $A$, then $C$ contains no more than $en/3$ columns. We can modify $M$ to make it $F$-free as follows: first modify every row that lies in a large cluster $R \in R$ to be equal to $r(R)$. Then pick some row $r$ of $M$ and modify all rows that are not contained in large clusters to be equal to $r$. Finally do the following: as long as $C$ is not empty, pick a column $c \in C$ that has a neighbour (predecessor or successor) not in $C$ and modify $c$ to be equal to its neighbour, and then remove $c$ from $C$.

It is not hard to see that since $F$ is unfoldable and closed under row permutations, after these modifications $M$ is $F$-free. Indeed, after the first and the second steps, all rows of $M$ are equal to rows from $Q$; the order of the rows does not matter since $F$ is closed under row permutations. Now each time that we modify a column $c \in C$ in the third step, all copies of matrices from $F$ that intersect it are destroyed and no new copies are created. By the maximality of $A$, any copy of a matrix from $F$ in the original $Q$ intersected some column from $C$, so we are done. The number of entry modifications needed in the first, second, third step respectively is at most $en^2/6, en^2/3, en^2/3$ and thus by making only $5en^2/6$ modifications of entries of $M$ we can make it $F$-free, contradicting the fact that $d(F, M) \geq \epsilon$.

Let $Q$ be any matrix of representatives of the large row clusters as above. Then $Q$ contains a collection $A$ of $en/3k$ pairwise disjoint copies of matrices from $F$. In particular, there exist a certain $s \times n$ submatrix $T$ of $Q$ and an $s \times t$ matrix $A(Q) \in F$ such that at least $en/3k |F|^s$ of the copies in $A$ are $A$-copies that lie in $T$. The following elementary removal lemma implies that $T$ contains many $A$-copies. The lemma will also be useful in Section 4.

**Lemma 2.4.** Fix an $s \times t$ matrix $A$. For any $\epsilon > 0$ there exists $\delta > 0$ such that if an $s \times n$ matrix $T$ satisfies $h(A, T) \geq \epsilon$ then $t(A, T) \geq \delta$, with $\delta^{-1}$ polynomial in $\epsilon^{-1}$.

**Proof.** Let $\epsilon > 0$. Let $T$ be a large enough $s \times n$ matrix with $h(A, T) \geq \epsilon$. There exists a collection $A$ of $en$ pairwise disjoint copies of $A$ in $T$. We construct $t$ disjoint subcollections $A_1, \ldots, A_t$ of $A$, each of size $en/2t \leq \lfloor en/t \rfloor$, such that for any $i < j$, all copies in $A_i$ are $i$-column-smaller than all copies in $A_j$. This is done by the following process for $i = 1, \ldots, t$: take $A_i$ to be the set of the $en/2t$ $i$-smallest copies in $A$ and delete these copies from $A$. Now observe that any $s \times t$ submatrix of $T$ that takes its $i$-th column (for $i = 1, \ldots, t$) as the $i$-th column of some copy from $A_i$ is equal to $A$. There are $(en/2t)^t$ such submatrices among all $\binom{n}{t} \leq n^t$ $s \times t$ submatrices of $T$, and so $t(A, T) \geq (\epsilon/2t)^t$. □

Lemma 2.4 implies that for $Q$ and $A(Q)$ as above, $t(F, Q) \geq t(A, Q) \geq \gamma$ where $\gamma^{-1}$ is polynomial in $(3k|F|r^s)^{-1}$ and so in $\epsilon^{-1}$. Finally we will show that $t(F, M) \geq \delta$ where $\delta^{-1}$ is polynomial in $\gamma^{-1}$ and so in $\epsilon^{-1}$, finishing the proof of the Theorem. For any large cluster $R \in R$ let $R'$ be some subcluster that contains exactly $\lfloor en/6r \rfloor > 0$ rows. Let $R' = \{R' : R \in R \text{ is large}\}$ and note that an $\alpha$-fraction of the $k \times k$ submatrices $S$ of $M$ have all of their rows in subclusters from $R'$ with no subcluster containing more than one row of $S$, where $\alpha^{-1}$ is polynomial in $\epsilon^{-1}$. Let $S$ be a random $k \times k$ submatrix of $M$. Conditioning on the event that $S$ satisfies the above property, we can assume that $S$ is chosen in the following way: first a
random $Q$ is created by picking uniformly at random one representative from every $R' \in \mathcal{R}'$, and then $S$ is taken as a random $k \times k$ submatrix of $Q$. As we have seen, there exists some $A(Q) \in F$ for which $t(A, Q) \geq \gamma$, implying that the probability that $S$ contains a copy of $A$ is at least $\gamma$. That is, a random $k \times k$ submatrix $S$ of $M$ contains a copy of a matrix from $F$ with probability at least $\alpha \gamma$, so there exists an $s \times t$ matrix $A \in F$ that is contained in a randomly chosen such $S$ with probability at least $\alpha \gamma/|F|$. Then $t(F, M) \geq t(A, M) \geq \delta$ where $\delta = \alpha \gamma/|F|^{k^{2k}}$: to see this, observe that we can choose a random $s \times t$ submatrix $S'$ of $M$ by first picking a random $k \times k$ submatrix $S$ and then picking an $s \times t$ random submatrix $S'$ of $S$. The event that $S$ contains a copy of $A$ has probability at least $\alpha \gamma/|F|$, and conditioned on this event, $S'$ is equal to $A$ with probability at least $k^{-2k}$, as the number of $s \times t$ submatrices of $S$ is at most $s^k t^k \leq k^{2k}$. This finishes the proof.

\[ \square \]

3 Proofs for the Non-Binary Case

We start with the (simple) proof of Proposition 1.6. The proof uses the (non induced) graph removal lemma. Some definitions are required for the proof. An $s \times t$ reordering $\sigma$ is a permutation of $[s] \times [t]$ that is a cartesian product of two permutations $\sigma_1 : [s] \to [s]$ and $\sigma_2 : [t] \to [t]$. Given an $s \times t$ matrix $A$, the $s \times t$ matrix $\sigma(A)$ is the result of the following procedure: first reorder the rows of $A$ according to the permutation $\sigma_1$ and then reorder the columns of the resulting matrix according to the permutation $\sigma_2$.

**Proof of Proposition 1.6.** Let $k(F)$ denote the largest row or column dimension of matrices from $F$. Let $\epsilon > 0$ and let $M$ be an $n \times n$ matrix over $\Gamma$ with index sets $[n], [n]$ that satisfies $h(F, M) \geq \epsilon$. Then there exists an $s \times t$ matrix $A \in F$ for which $h(A, M) \geq \epsilon/|F|$. Since $F$ is closed under permutations, $\sigma(F) \subseteq F$ for any $s \times t$ reordering $\sigma$, so $t(F, M) \geq \sum_{\sigma} t(\sigma(A), M)$ where $\sigma$ in the sum ranges over all possible $s \times t$ reorderings.

We construct an $(s+t)$-partite graph $G$ on $(s+t)n$ vertices as follows: there are $s$ row parts $R_1, \ldots, R_s$ and $t$ column parts $T_1, \ldots, T_t$, each containing $n$ vertices. The vertices of $R_i$ ($C_i$) are labelled $r_1^i, \ldots, r_n^i$ ($c_1^i, \ldots, c_n^i$ respectively). Two vertices $r_i^a$ and $r_j^b$ (or $c_i^a$ and $c_j^b$) with $a \neq b$ are connected by an edge iff $i \neq j$. $r_i^a$ and $c_j^b$ are connected iff $M(i,j) = A(a,b)$.

We now show that there exists a bijection between copies of $K_{s+t}$ in $G$ and couples $(S, \sigma)$ where $S$ is an $s \times t$ submatrix of $M$ and $\sigma$ is an $s \times t$ reordering such that $\sigma(S) = A$. Indeed, take the following mapping: a couple $(S, \sigma)$, where $S$ is the submatrix of $M$ on $\{a_1, \ldots, a_s\} \times \{b_1, \ldots, b_t\}$ with $a_1 < \ldots < a_s$ and $b_1 < \ldots < b_t$ and $\sigma = \sigma_1 \times \sigma_2$, is mapped to the induced subgraph of $G$ on $\{r_{\sigma_1(1)}^{a_1}, \ldots, r_{\sigma_1(s)}^{a_s}, c_{\sigma_2(1)}^{b_1}, \ldots, c_{\sigma_2(t)}^{b_t}\}$.

It is not hard to see that $(S, \sigma)$ is mapped to a copy of $K_{s+t}$ if and only if $\sigma(S)$ is equal to $A$. On the other hand, every copy of $K_{s+t}$ in $G$ has exactly one vertex in each row part and in each column part, and there exists a unique couple $(S, \sigma)$ mapped to it.

Since $h(A, M) \geq \epsilon/|F|$ there exist $\epsilon n^2/|F| k^{2k}$ disjoint $A$-copies in $M$ that are mapped (with the identity reordering) to edge-disjoint copies of $K_{s+t}$ in $G$. By the graph removal lemma, there exists $\delta > 0$ such that at least a $\delta$-fraction of the subgraphs of $G$ on $s+t$ vertices are cliques. Therefore, at least a $\delta$-fraction of the possible couples $(S, \sigma)$ (where $S$ is an $s \times t$ submatrix of $M$ and $\sigma$ is an $s \times t$ reordering) satisfy $\sigma(S) = A$, implying that $t(F, M) \geq \sum_{\sigma} t(\sigma(A), M) \geq \delta$ as desired.

\[ \square \]

Next we give the proof of Theorem 1.8. We may and will assume throughout the proof that $M$ is an $n \times n$ matrix where $n$ is large enough with respect to $\epsilon$.

**Proof of Theorem 1.8.** We start with deriving Statement 2 from Statement 1; this direction is quite straightforward, while the other direction is more interesting. Fix an $s \times t$ matrix $A$. Let $\epsilon > 0$ and assume that Statement 1 holds. There exists $\delta = \delta(\epsilon)$ for which $t(A, M) < \delta$ implies $h(A, M) < \epsilon$ for any $n \times n$ matrix $M$. To prove Statement 2 we can pick $\delta' = \delta'(\epsilon) > 0$
small enough such that for any large enough \( n \times n \) matrix \( M \), any \( 1 \leq i \leq s - 1 \) and any \( 1 \leq j \leq t - 1 \), the fraction of \( s \times t \) submatrices with \( i \)-height (or \( j \)-width) smaller than \( \delta \) among all \( s \times t \) submatrices is at most \( \delta/2 \). Thus, any \( M \) for which there exists some \( 1 \leq i \leq s - 1 \) (or \( 1 \leq j \leq t - 1 \)) such that all \( A \)-copies in \( M \) have \( i \)-height (\( j \)-width) smaller than \( \delta \) satisfies \( t(A, M) \leq \delta/2 < \delta \), implying that \( h(A, M) < \delta \) as desired.

Next we assume that Statement 2 holds and prove Statement 1. Fix an \( s \times t \) matrix \( A \) over an alphabet \( \Gamma \), let \( \epsilon > 0 \) and let \( M \) be a large enough \( n \times n \) matrix with \( h(A, M) \geq \epsilon \). We will show that there exist \( \epsilon' > 0 \) that depends only on \( \epsilon \), sets \( X, Y \) of row and column separators respectively of sizes \( s - 1 \) and \( t - 1 \) and a collection of \( \epsilon' n^2 \) disjoint \( A \)-copies separated by \( X \times Y \) in \( M \). Then we will combine a simpler variant of the construction used in the proof of Proposition 1.6 with the graph removal lemma to show that \( t(A, M) \geq \delta \) for a suitable \( \delta(\epsilon) > 0 \).

Since \( h(A, M) \geq \epsilon \), \( M \) contains a maximal collection \( A_0 \) of at least \( \epsilon n^2 \) pairwise disjoint copies of \( A \) where \( \epsilon_0 = \epsilon/\delta \) (as the set of entries of \( A \)-copies in \( A_0 \) intersects all \( A \)-copies in \( M \)). The quantity \( t(A, M) \) does not depend on the alphabet, so we may consider \( A \) and \( M \) as matrices over the alphabet \( \Gamma \), respectively of sizes \( s - 1 \) and \( t - 1 \) and a collection of \( \epsilon' n^2 \) disjoint \( A \)-copies separated by \( X \times Y \) in \( M \). Therefore, the expected number of \( A \)-copies from \( \Gamma \), even though all symbols in \( A \) and \( M \) are from \( \Gamma \). Without loss of generality we will assume that no two entries in \( A \) are equal; as we shall see, a \( \delta(\epsilon) \)-fraction of the \( s \times t \) submatrices of \( M \) are \( A \)-copies that take all of their entries from \( A_0 \), independently of the structure of \( A_0 \), so this assumption is valid.

Let \( M_0 \) be the following \( n \times n \) matrix over \( \Gamma' \): all \( A \)-copies in \( A_0 \) appear in the same locations in \( M_0 \), and all other entries of \( M_0 \) are equal to \( \alpha \). Clearly \( t(A, M_0) \leq t(A, M) \), as all \( A \)-copies in \( M_0 \) also exist in \( M \). Also let \( X_0 = \phi \). Next, we construct iteratively for any \( i = 1, \ldots, s - 1 \) an \( n \times n \) matrix \( M_i \) over \( \Gamma' \) that contains a collection \( A_i \) of \( \epsilon_i n^2 \) pairwise disjoint copies of \( A \) where \( \epsilon_i > 0 \) depends only on \( \epsilon_{i-1} \), such that all \( A \)-copies in \( M_i \) also exist in \( M_{i-1} \), and so \( t(A, M_i) \leq t(A, M_{i-1}) \). We also maintain a set \( X_i \) of row separators whose elements are \( x_1 < \ldots < x_i \), such that any entry of \( M \) between \( x_{j-1} \) and \( x_j \) for \( j = 1, \ldots, i \) (where we define \( x_0 = 0, x_i = n \)) is either equal to one of the entries of the \( j \)-th row of \( A \) or to \( \alpha \).

The construction of \( M_i \) given \( M_{i-1} \) is done as follows. By Statement 2, there exists \( \delta_i = \delta_i(\epsilon_{i-1}) \) such that any matrix \( M' \) over \( \Gamma' \) with \( h(A, M') \geq \epsilon_{i-1}/2 \) contains a copy of \( A \) with \( i \)-height at least \( \delta_i \). We start with a matrix \( M' \) equal to \( M_{i-1} \) and an empty \( A_i \), and as long as \( M' \) contains a copy of \( A \) with \( i \)-height at least \( \delta_i \), we add it to \( A_i \) and modify (in \( M' \)) all entries of all \( A \)-copies from \( A_{i-1} \) that intersect it to \( \alpha \). By the separation that \( X_{i-1} \) induces on \( M' \), each such copy has its \( j \)-th row between \( x_{j-1} \) and \( x_j \) for any \( 1 \leq j \leq i - 1 \).

This process might stop only when \( h(A, M') < \epsilon_{i-1}/2 \), and in each step only \( st \) copies of \( A \) are deleted from \( M' \), so in the end \( \mathcal{A}_i \) contains at least \( \epsilon_{i-1} n^2/2st \) pairwise disjoint copies of \( A \) with \( i \)-height at least \( \delta_i \). Pick uniformly at random a row index \( x_i > x_{i-1} \). The probability that a certain copy of \( A \) in \( \mathcal{A}_i \) has its \( i \)-th row at or above \( x_i \) and its \((i + 1)\)-th row below \( x_i \) is at least \( \delta_i \). Therefore, the expected number of \( A \)-copies in \( \mathcal{A}_i \) with this property is at least \( \epsilon_i n^2 \) with \( \epsilon_i = \delta_i \epsilon_{i-1}/2st \), so there exists some \( x_i \) such that at least \( \epsilon_i n^2 \) \( A \)-copies in \( \mathcal{A}_i \) have their first \( i + 1 \) rows separated by \( X_i = X_{i-1} \cup \{x_i\} \); delete all other copies from \( \mathcal{A}_i \). We now construct \( M_i \) as follows: all \( A \)-copies from \( \mathcal{A}_i \) appear in the same locations in \( M_i \), and all other entries of \( M_i \) are equal to \( \alpha \). After iteration \( s - 1 \) we have a matrix \( M_{s-1} \) with \( \epsilon_{s-1} n^2 \) copies of \( A \) separated by \( X = X_{s-1} \). We apply the same process in columns instead of rows, starting with the matrix \( M_{s-1} \). The resulting matrix \( M^* \) contains \( \epsilon^* n^2 \) pairwise disjoint copy of \( A \) separated by \( X \times Y \) of the column separators \( y_1 < \ldots < y_{t-1} \), \( \epsilon^* \) depends on \( \epsilon \) (but not on \( n \)) and \( t(A, M^*) \leq t(A, M_{s-1}) \leq t(A, M) \).

Finally, construct an \((s + t)\)-partite graph \( G \) on \( 2n \) vertices as follows: the row parts are \( R_1, \ldots, R_s \) and the column parts are \( C_1, \ldots, C_t \) where \( R_i \) (\( C_i \)) contains vertices labelled \( x_{i-1} + 1, \ldots, x_i \) \((y_{i-1} + 1, \ldots, y_i)\) respectively with \( x_0 = y_0 = 0, x_n = y_t = n \). Any two row (column) vertices not in the same part are connected. Vertices \( a \in R_i, b \in C_j \) are connected if and only if \( M^*(a, b) = A(i, j) \). Clearly there exists a bijection between \( A \)-copies in \( M^* \) and \( K_{s+t} \) copies in \( G \) that maps disjoint \( A \)-copies to edge disjoint \( K_{s+t} \)-copies in \( G \), so it contains \( \epsilon^* n^2 \) edge
disjoint \((s+t)\)-cliques. By the graph removal lemma there exists \(\delta = \delta(e^*) > 0\) such that a \(\delta\)-fraction of the subgraphs of \(G\) on \(s + t\) vertices are cliques. Hence \(t(A, M) \geq (A, M^*) \geq \delta\), finishing the proof.

\[\square\]

### 3.1 Lower bound

In this subsection we give an alternative constructive proof of Theorem 1.5. Our main tool is the following result in additive number theory from [1], based on a construction of Behrend [6].

**Lemma 3.1** [1, 6]. For every positive integer \(m\) there exists a subset \(X \subseteq [m] = \{1, \ldots, m\}\) with no non-trivial solution to the equation \(x_1 + x_2 + x_3 = 3x_4\), where \(X\) is of size at least

\[
|X| \geq \frac{m}{e^{20 \sqrt{\log m}}}. \tag{1}
\]

**Proof of Theorem 1.5.** Let

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Let \(m\) be a positive integer divisible by 10 and let \(X \subseteq [m/10]\) be a subset with no non-trivial solution to the equation \(x_1 + x_2 + x_3 = 3x_4\) that is of maximal size. We construct the following \(m \times m\) ternary matrix \(M\). For any \(1 \leq i \leq m/5\) and any \(x \in X\) we put a copy of \(A\) in \(M\) as follows:

\[
M(i, i + x) = M(m/2 + i + 2x, m/2 + i + 3x) = 1
\]

\[
M(i, m/2 + i + 3x) = M(m/2 + i + 2x, i + x) = 0.
\]

We set all other entries of \(M\) to 2. Let \(A\) be the collection of \(q = m|X|/5 \geq m^2/50e^{20 \sqrt{\log m}}\) pairwise disjoint copies of \(A\) in \(M\) that were created as above. Note that all \(A\)-copies in \(M\) are separated by \(\{n/2\} \times \{n/2\}\), where there are two opposite quarters (with respect to the separation) that do not contain the entry 0 and the two other opposite quarters do not contain 1. Hence, every \(A\)-copy must contain one entry from each quarter. The main observation is that all of the \(A\)-copies in \(M\) are actually copies from \(A\), so \(M\) contains exactly \(q A\)-copies.

To see this, suppose that the rows of an \(A\)-copy in \(M\) are \(i\) and \(j + n/2\) for some \(1 \leq i, j \leq n/2\), then there exist \(x_1, x_2, x_3, x_4 \in X\) such that the entries of the copy were taken from locations \((i, i + x_1), (i, m/2 + i + 3x_2), (m/2 + j, j - x_3), (m/2 + j, m/2 + j + x_4)\) in \(M\) and so we have \(i + x_1 = j - x_3\) and \(i + 3x_2 = j + x_4\). Reordering these two equations we get that \(3x_2 = x_1 + x_3 + x_4\), implying that \(x_1 = x_2 = x_3 = x_4\) and \(j = i + 2x_1\), so the above \(A\)-copy is indeed in \(A\).

Let \(n\) be an arbitrarily large positive integer divisible by \(m\). Given \(M\) as above, we create an \(n \times n\) ‘blowup’ matrix \(N\) as follows: for any \(1 \leq i, j \leq n\), \(N(i, j) = M(\lfloor im/n \rfloor, \lfloor jm/n \rfloor)\). \(N\) can also be seen as the result of replacing any entry \(e\) in \(M\) with an \(n/m \times n/m\) matrix of entries equal to \(e\). The total number of \(A\)-copies in \(N\) is exactly \((n/m)^4 q = n^4|X|/5m^3\), whereas the maximum number of pairwise disjoint \(A\)-copies in \(N\) is exactly \((n/m)^2 q = n^2|X|/5m\). Assuming that \(\epsilon > 0\) is small enough and picking \(m\) to be the smallest integer divisible by 10 and larger than \((c/\epsilon)^{c \log(c/\epsilon)}\) for a suitable absolute constant \(c > 0\) gives that \(h(A, N) = |X|/5m > \epsilon\) and \(t(A, N) = n^4|X|/5m^3\) is \(2 < 1/m^2 < (c/\epsilon)^{-c \log(c/\epsilon)}\) as needed. Note that the statement of the theorem requires that we find a family \(F\) that is closed under permutations, not a single matrix \(A\). The family \(F = \{A, B\}\) where

\[
B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

is closed under permutations, and since \(N\) is \(B\)-free, we get that \(h(F, N) > \epsilon\) and \(t(F, N) < (c/\epsilon)^{-c \log(c/\epsilon)}\) as desired.

\[\square\]
4 Foldable and Unfoldable Matrices

This short section is dedicated to the proof of Lemma 1.9. Our main lemma in this section, which implies Lemma 1.9, is the following.

**Lemma 4.1.** Let $A$ be a fixed matrix that has two equal neighbouring rows (or columns) and let $A'$ be the matrix created from $A$ by deleting one of these rows (columns). Then for any $\epsilon > 0$ there exist $n^* = n^*(A, \epsilon) > 0$ and $\tau = \tau(A, \epsilon) > 0$ such that for any $n \geq n^*$, any $n \times n$ matrix $M$ with $t(A', M) \geq \epsilon$ also satisfies $t(A, M) \geq \tau$. Moreover, $n^*$ and $\tau^{-1}$ are polynomial in $\epsilon^{-1}$.

To see why Lemma 1.9 follows from Lemma 4.1, observe that there exists a sequence $A = A_0, A_1, \ldots, A_k = A$ of matrices such that $A_{i+1}$ is created from $A_i$ by deleting either one of the rows of $A_i$ that is equal to its predecessor or one of the columns of $A_i$ that is equal to its predecessor. Now let $\epsilon > 0$, denoting $\tau_k = \epsilon$ and apply Lemma 4.1 iteratively. For any $i = k - 1, k - 2, \ldots, 0$ let $\tau_i = \tau(A_{i+1}, \tau_{i+1})$ and $n_i = n^*(A_{i+1}, \tau_{i+1})$ be the quantities in the statement of Lemma 4.1 that are obtained when applying the lemma on $A_i$. Since $\tau_{k-1}^{-1}, n_i$ are polynomial in $\tau_{k-1}^{-1}$, an easy induction implies that $\tau_{k-1}^{-1}$ and $n_i$ are polynomial in $\epsilon^{-1}$ for any $i = k - 1, \ldots, 0$. Hence the statement of Lemma 1.9 holds for $n^0 = \max\{n_0, \ldots, n_{k-1}\}$ and $\delta = \tau_0$, where $n^0$ and $\delta^{-1}$ are polynomial in $\epsilon^{-1}$.

**Proof of Lemma 4.1.** Suppose that the dimensions of $A$ are $(s + 1) \times t$, so $A'$ has dimensions $s \times t$. Let $1 \leq j \leq s$ be the index of a row in $A$ that is equal to its successor and can be deleted from $A$ to create $A'$. Let $\epsilon > 0$ and let $M$ be an $n \times n$ matrix with $t(A', M) \geq \epsilon$ and $n \geq n^* = \max\{10s^2/\epsilon, t\}$. Since $t(A', M) \geq \epsilon$, there are at least $\epsilon(n^*)(n^*)$ copies of $A'$ in $M$. Let $T$ be the family of all $n \times t$ submatrices $S$ of $M$ containing at least $\epsilon(n^*)/2$ copies of $A'$. Any $S \in T$ has $(n^*)$ $s \times t$ submatrices, so the number of $A'$ copies in submatrices from $T$ is at most $|T|(n^*)$. On the other hand, there are $(n^*)$ $n \times t$ submatrices of $M$ so the number of $A'$ copies in $n \times t$ submatrices not in $T$ is less than $(n^*)^2\epsilon(n^*)/2$. Hence the total number of $A'$ copies in submatrices from $T$ is at least $\epsilon(n^*)(n^*)/2$, implying that $|T| \geq \epsilon(n^*)/2$.

Let $S \in T$. $S$ contains a collection $A(S)$ of $en^2/2s^2$ pairwise disjoint copies of $A'$. To show this, we follow a greedy approach, starting with a collection $S'$ of all $A'$-copies in $S$ and with empty $A$. As long as $S'$ is not empty, we arbitrarily choose a copy $C' \in S'$ of $A'$, add $C'$ to $A$ and delete all $A'$-copies intersecting $C'$ (including itself) from $S'$. In each step, the number of deleted copies is at most $s(n^*-s) \leq s^2(n^*)/n$, so the number of steps is at least $\epsilon(n^*)n^2/2s^2(n^*) = en^2/2s^2$.

Let $\delta_1 = \epsilon/5s^2$. Assuming that $n$ is large enough, there exist a row $r$ in $S$ and disjoint subsets $A_1, A_2 \subseteq A$ of size $\delta_1 n$ each, such that all $A'$-copies in $A_1$ ($A_2$) are $j$-smaller ($j$-larger respectively) than $r$. Let $C$ be a set of $\delta_1 n$ disjoint couples of $A'$-copies $(C_1, C_2)$ where $C_1 \subseteq A_1$, $C_2 \subseteq A_2$. From every couple in $C$ we can create a copy of $A$ as follows: pick rows $1, \ldots, j$ of $C_1$ and rows $j, \ldots, s$ of $C_2$ and concatenate them to get an $(s + 1) \times t$ matrix equal to $A$. Therefore, $S$ contains $\delta_1 n$ pairwise disjoint $A$-copies and so $h(A, S) \geq \delta_1 n/nt = \epsilon/5s^2t$. By Lemma 2.4, $t(A, S) \geq \delta$ with $\delta^{-1}$ polynomial in $\epsilon^{-1}$. Therefore, there are at least $\delta(n^*)|T| \geq \delta\epsilon(n^*)(n^*)/2$ copies of $A$ in $M$. Taking $\tau = \delta\epsilon/2$ finishes the proof.

5 Concluding Remarks

Generally, property testing seems to be easier for objects that are highly symmetric. A good example of this phenomenon is the problem of testing properties of (ordered) one-dimensional binary vectors. There are some results on this subject, but it is far from being well understood. On the other hand, the binary vector properties $P$ that are invariant under permutations of the entries (these are the properties in which for any vector $v$ that satisfies $P$, any permutation of the entries of $v$ also satisfies $P$) are merely those that depend only on the length and the Hamming weight of a vector. This makes the task of testing these properties trivial.
A central example of the symmetry phenomenon is the well investigated subject of property testing in graphs, that considers only functions from $\binom{[n]}{2}$ to $\{0,1\}$ that are invariant under permutations of $\binom{[n]}{2}$ induced by permutations on $[n]$. That is, if a labeled graph $G$ satisfies some graph property, then any relabeling of its vertices results in a graph that also satisfies this property. See [21] for further discussion on the role of symmetries in property testing.

In general, matrices (with row and column order) do not have any symmetries. Therefore, the above reasoning suggests that proving results on the testability of matrix properties is likely to be harder than proving similar results on properties of matrices where only the rows are ordered (such properties are invariant under permutations of the columns), which might be harder in turn than proving the same results for properties of matrices without row and column orders, i.e. bipartite graphs, as these properties are invariant under permutations of both the rows and the columns.

Theorem 1.1 is a weak removal lemma for binary matrices with row and column orders, while Theorem 1.2 is a strong removal lemma for binary matrices without row and column orders and our generalization of it, Theorem 1.4, is a strong removal lemma for binary matrices with a row order but without a column order. It will be very interesting to settle Problem 1.3, that asks whether a strong removal lemma exists for binary matrices with row and column orders.

It will be interesting to expand our knowledge of matrices in two and higher dimensions and of ordered combinatorial objects in general. Note that some of the results in this paper are also true for high dimensional matrices: Proposition 1.6, Theorem 1.8 and Lemma 1.9 are easily generalizable to matrices in more than two dimensions, and their proofs are essentially identical. The main difference in the proofs of Proposition 1.6 and Theorem 1.8 is that we make use of the hypergraph removal lemma [14, 17, 18, 23] instead of the graph removal lemma. The lower bound in Theorem 1.5 also holds for three-dimensional binary matrices as was observed in [10], implying that there cannot be three-dimensional variants of Theorems 1.1 and 1.4 with $\delta^{-1}$ that is polynomial in $\epsilon^{-1}$.

In the non-binary case, Proposition 1.6 gives a weak removal lemma for matrices without row and column orders. It will be interesting to get results of this type for less symmetric objects, ultimately for matrices with row and column orders. Theorem 1.8 implies that solving the following seemingly innocent problem, which is a special case of Problem 1.7, will settle it for the case that $A$ is a $2 \times 2$ matrix, and the techniques used to solve it might help in settling Problem 1.7 in general. We say that the 1-height of a $2 \times 2$ matrix is simply its height.

**Problem 5.1.** Let $A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$. Is it true that for any $\epsilon > 0$ there exists $\delta > 0$ such that any matrix $M$ (over any alphabet containing 0, 1, 2, 3) in which all $A$-copies have height less than $\delta$ satisfies $h(A, M) < \epsilon$?

As a final remark, in the results in which $\delta^{-1}$ is polynomial in $\epsilon^{-1}$ we have not tried to obtain tight bounds on the dependence, and it may be interesting to do so.

**References**


