Many T copies in H-free graphs

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Abstract

For two graphs T and H with no isolated vertices and for an integer n, let $ex(n, T, H)$ denote the maximum possible number of copies of T in an H-free graph on n vertices. The study of this function when $T = K_2$ is a single edge is the main subject of extremal graph theory. In the present paper we investigate the general function, focusing on the cases of triangles, complete graphs, complete bipartite graphs and trees. These cases reveal several interesting phenomena. Three representative results are:

(i) $ex(n, K_3, C_5) \leq (1 + o(1))\frac{\sqrt{3}}{2} n^{3/2}$,

(ii) For any fixed m, $s \geq 2m - 2$ and $t \geq (s-1)! + 1$, $ex(n, K_m, K_{s,t}) = \Theta(n^{m - \binom{m}{2}/s})$ and

(iii) For any two trees H and T, $ex(n, T, H) = \Theta(n^m)$ where $m = m(T, H)$ is an integer depending on H and T (its precise definition is given in Section 1).

The first result improves (slightly) an estimate of Bollobás and Győri. The proofs combine combinatorial and probabilistic arguments with simple spectral techniques.

1   Introduction

For two graphs T and H with no isolated vertices and for an integer n, let $ex(n, T, H)$ denote the maximum possible number of copies of T in an H-free graph on n vertices.

When $T = K_2$ is a single edge, $ex(n, T, H)$ is the well studied function, usually denoted by $ex(n, H)$, specifying the maximum possible number of edges in an H-free graph on n vertices. There is a huge literature investigating this function, starting with the theorems of Mantel [34] and Turán [42] that determine it for $H = K_r$. See, for example, [39] for a survey.

In the present paper we show that the function for other graphs T besides K_2 exhibits several additional interesting features. We illustrate these by presenting several general results and by focusing on various special cases of graphs H and T in certain prescribed families. The question is interesting for many other graphs T and H, and many of the results here can be extended.

There are several sporadic papers dealing with the function $ex(n, T, H)$ for $T \neq K_2$. The first one is due to Erdős in [16], where he determines $ex(n, K_t, K_r)$ for all $t < r$ (see also [9] for an extension).

A notable recent example is given in [27], where the authors determine this function precisely for $T = C_5$ and $H = K_3$.

The case $T = K_3$ and $H = C_{2k+1}$ has also been studied. Bollobás and Győri [8] proved that

$$(1 + o(1))\frac{1}{3\sqrt{3}}n^{3/2} \le ex(n, K_3, C_5) \le (1 + o(1))\frac{5}{4}n^{3/2}. \tag{1}$$

Győri and Li [25] proved that for any fixed $k \ge 2$

$$\binom{k}{2}ex_{bip}(\frac{2n}{k+1}, C_4, C_6, \ldots, C_{2k}) \le ex(n, K_3, C_{2k+1}) \le \frac{(2k-1)(16k-2)}{3}ex(n, C_{2k}), \tag{2}$$

where $ex_{bip}(m, C_4, C_6, \ldots, C_{2k})$ denotes the maximum possible number of edges in a bipartite graph on m vertices and girth exceeding $2k$.

Here we start with a simple characterization of the pairs of graphs H and T for which $ex(n, T, H) = \Theta(n^t)$, where t is the number of vertices of T. Combining this observation with the graph removal lemma we establish an Erdős-Stone type result by giving an asymptotic formula for $ex(n, K_t, H)$ for any graph H with chromatic number $\chi(H) > t$.

Next we study the case $T = K_3$. Our first result characterizes all graphs H for which $ex(n, K_3, H) \le c(H)n$. The friendship graph F_k is the graph consisting of k triangles with a common vertex. Equivalently, this is the graph obtained by joining a vertex to all $2k$ vertices of a matching of size k. Call a graph an extended friendship graph iff its 2-core is either empty or F_k for some positive k.

Theorem 1.1. *There exists a constant $c(H)$ so that $ex(n, K_3, H) \le c(H)n$ if and only if H is a subgraph of an extended friendship graph.*

We also slightly improve the upper estimates in (1) and in (2) above, proving the following.

Proposition 1.1. *The following upper bounds hold.*
(i) $ex(n, K_3, C_5) \le (1 + o(1))\frac{\sqrt{3}}{2}n^{3/2}$.
(ii) For any $k \ge 2$, $ex(n, K_3, C_{2k+1}) \le \frac{16(k-1)}{3} ex(\lceil n/2 \rceil, C_{2k})$.

A similar result has been proved independently by Füredi and Özkahya [23], who showed that $ex(n, K_3, C_{2k+1}) \le 9k \, ex(n, C_{2k})$.

The next theorem deals with maximizing the number of copies of a complete graph while avoiding complete bipartite graphs:

Theorem 1.2. *For any fixed m and $t \ge s$ satisfying $s \ge 2m - 2$ and $t \ge (s - 1)! + 1$ there are two constants $c_1 = c_1(s, t)$ and $c_2 = c_2(s, t)$ such that*

$$c_1 n^{m - \binom{m}{2}/s} \le ex(n, K_m, K_{s,t}) \le c_2 n^{m - \binom{m}{2}/s}.$$

The last two theorems focus on the case where the excluded graph H is a tree. Before stating the results we give the following definitions:

Definition 1.3. *For a graph T, a set of vertices $U \subseteq V(T)$ and an integer h, the (U, h) blow-up of T is the following graph. Fix the vertices in U, and replace each connected component in $T \setminus U$ with h vertex disjoint copies of it connected to the vertices of U exactly as the original component is connected to these in T.*

Definition 1.4. *For two trees, T and H, let $m(T, H)$ be the maximum integer m such that there is a $(U, |V(H)|)$ blow-up of T containing no copy of H and having m connected components in $T \setminus U$.*

In this notation we prove the following.

Theorem 1.5. *For every two integers t and h there are positive constants $c_1(t, h), c_2(t, h)$ so that the following holds. Let H be a tree on h vertices and let T be a tree on t vertices, then*

$$c_1(t, h)n^m \leq ex(n, T, H) \leq c_2(t, h)n^m,$$

where $m = m(T, H)$.

Finally we consider the case where T is a bipartite graph and H is a tree. For a tree H, any H-free graph can have at most a linear number of edges. Therefore, by a theorem proved in [1], the maximum possible number of copies of any bipartite graph T in it is bounded by $O(n^{\alpha(T)})$, where $\alpha(T)$ is the size of a maximum independent set in T. Using the next definition we characterize the cases in which $ex(n, T, H) = \Theta(n^{\alpha(T)})$.

Definition 1.6. *An edge cover of a graph T (with no isolated vertices) is a set $\Gamma \subset E(T)$ such that for each vertex $v \in V(T)$ there is an edge $e \in \Gamma$ for which $v \in e$. Call an edge-cover minimum if it has the smallest possible number of edges.*

A set of vertices $U \subset V(T)$ is called a $U(\Gamma)$-set if each connected component of $T \setminus U$ intersects exactly one edge of Γ, and the number of these connected components is $|\Gamma|$.

Note that if Γ is an edge cover of T and U is a $U(\Gamma)$ set, then any connected component of $T \setminus U$ is either an edge of Γ or a single vertex.

Theorem 1.7. *Let T be a bipartite graph with no isolated vertices and let H be a tree on h vertices. Then the following are equivalent:*

1. *$ex(n, T, H) = \Theta(n^{\alpha(T)})$*

2. *For any minimum edge-cover Γ of T there is a choice of a $U(\Gamma)$-set U such that the (U, h) blow-up of T does not contain a copy of H,*

3. *For some minimum edge cover Γ of T there is a choice of a $U(\Gamma)$-set U such that the (U, h) blow-up of T does not contain a copy of H.*

It is worth noting that for $T \neq K_2$ the function $ex(n, T, H)$ behaves very differently from its well studied relative $ex(n, H) = ex(n, K_2, H)$. In particular, it is easy to see that for any graph H with at least 2 edges, if $2H$ denotes the vertex disjoint union of two copies of H, then $ex(n, H)$

and $ex(n, 2H)$ have the same order of magnitude. In contrast, if, for example, $H = C_5$ then by (1), $ex(n, K_3, H) = \Theta(n^{3/2})$ and it is not difficult to show that $ex(n, K_3, 2H) = \Theta(n^2)$. Similarly, it is known that for any graph H, $ex(n, H)$ is either quadratic in n or is at most $n^{2-\epsilon(H)}$ for some fixed $\epsilon(H) > 0$, whereas it is not difficult to deduce from the results of Ruzsa and Szemerédi in [37] that for the graph H consisting of two triangles sharing an edge $n^{2-o(1)} \leq ex(n, K_3, H) \leq o(n^2)$ as shown in Section 3.

The rest of this paper is organized as follows. In Section 2 we consider the dense case, describing the simple characterization of pairs of graphs T and H so that $ex(n, T, H) = \Theta(n^t)$ with t being the number of vertices of T, and establishing an Erdős-Stone type result for $ex(n, K_t, H)$ when $\chi(H) > t$. In Section 3 we study the case $T = K_3$, proving Theorem 1.1 and Proposition 1.1. The proof of Theorem 1.2 is given in Section 4, together with several related results, and the proofs of Theorems 1.5 and 1.7 are described in Section 5. The final Section 6 contains some concluding remarks and open problems.

2   The dense case

The case where both T and H are complete graphs is studied by Erdős in [16] where he determines that:

$$ex(n, K_t, K_k) = \sum_{0 \leq i_1 \leq \cdots \leq i_t \leq k-2} \prod_{r=1}^{t} \lfloor \frac{n + i_r}{k - 1} \rfloor.$$

A similar (though less accurate) result can be obtained for general graphs. We proceed with the simple details.

An s blow-up of a graph H is the graph obtained by replacing each vertex v of H by an independent set W_v of size s, and each edge uv of H by a complete bipartite graph between the corresponding two independent sets W_u and W_v.

As this is going to be useful throughout the paper, denote the number of copies of a fixed graph H in a graph G by $\mathcal{N}(G, H)$.

Proposition 2.1. *Let T be a fixed graph with t vertices. Then $ex(n, T, H) = \Omega(n^t)$ iff H is not a subgraph of a blow-up of T. Otherwise, $ex(n, T, H) \leq n^{t-\epsilon(T,H)}$ for some $\epsilon(T, H) > 0$.*

Proof. If H is not a subgraph of a blow-up of T then the graph G which is the $\ell = \lfloor n/t \rfloor$-blow-up of T contains no copy of H and yet includes at least $\ell^t = \Omega(n^t)$ copies of T. This establishes the first part of the proposition.

To prove the second part, assume that H is a subgraph of the s-blow-up of T. We have to show that in this case any H-free graph $G = (V, E)$ on n vertices contains less than $n^{t-\epsilon}$ copies of T for some $\epsilon = \epsilon(T, H) > 0$. Indeed, suppose that G contains m copies of T. Let $V = V_1 \cup V_2 \cup \cdots \cup V_t$ be a random partition of V into t pairwise disjoint classes. Let $u_1, u_2, \ldots, u_t$ denote the vertices of T. Then the expected number of copies of T in which u_i belongs to V_i for all i is m/t^t. Thus we can fix a partition $V = V_1 \cup V_2 \cup \cdots \cup V_t$ so that the number of such copies of T is at least m/t^t. Construct a t-uniform, t-partite hypergraph on the classes of vertices $V_1, \ldots, V_t$ by letting a set of

vertices $v_1, \ldots, v_t$ with $v_i \in V_i$ be an edge iff G contains a copy of T on these vertices, where v_i plays the role of u_i for each i. Therefore, this hypergraph contains at least m/t^t edges. By a well known result of Erdős [15], if the number of edges exceeds $n^{t-\epsilon}$ for an appropriate $\epsilon = \epsilon(t, s) > 0$, then this hypergraph contains a complete t-partite hypergraph with classes of vertices $U_i \subset V_i$, $|U_i| = s$ for all i. This gives an s-blow-up of T in the original graph G, providing a copy of H in it, contradiction. It follows that $m \leq t^t n^{t-\epsilon}$, completing the proof. $\qquad \square$

When $T = K_t$ is a complete graph, H is not a subgraph of any blow-up of $T = K_t$ if and only if $\chi(H) > t$. In this case it is not difficult to determine the asymptotic value of $ex(n, T, H)$ up to a lower order additive term, as we show next. Note that the case $t = 2$ is the classical result of Erdős and Stone [19].

Proposition 2.2. *For any graph H, $ex(n, K_t, H) = \Omega(n^t)$ if and only if $\chi(H) > t$. Furthermore, if indeed $\chi(H) = k > t$ then $ex(n, K_t, H) = (1 + o(1))\binom{k-1}{t}(\frac{n}{k-1})^t$*

Proof. The first part follows directly from Proposition 2.1. To prove the second part fix t and H, and suppose that $\chi(H) = k > t$. We have to show that $ex(n, K_t, H) = (1 + o(1))\binom{k-1}{t}(\frac{n}{k-1})^t$.

The lower bound is obtained by taking a Túran graph with no copy of K_k. To prove the upper bound, assume G is an H-free graph on n vertices satisfying $\mathcal{N}(G, K_t) = ex(n, K_t, H)$. By the previous proposition 2.1, as G is H-free $\mathcal{N}(G, K_k) \leq ex(n, K_k, H) = o(n^k)$.

Using the graph removal lemma (as stated in [2] following [37] and improved in [20]) we can remove $o(n^2)$ edges from G and get a new graph G' which is K_k-free. The removal of $o(n^2)$ edges from G can remove at most $o(n^2)O(n^{t-2}) = o(n^t)$ copies of K_t, thus $\mathcal{N}(G', K_t) = (1+o(1))\mathcal{N}(G, K_t)$. As G' is K_k free one has $\mathcal{N}(G', K_t) \leq ex(n, K_t, K_k) = \sum_{0 \leq i_1 \leq \cdots \leq i_t \leq k-2} \prod_{r=1}^{t} \lfloor \frac{n+i_r}{k-1} \rfloor$. This yields the needed result. $\qquad \square$

3   Maximizing the number of triangles

3.1   Extended friendship graphs

In this subsection we prove Theorem 1.1. Here and throughout the paper, we often do not make any serious attempt to optimize the absolute constants. We also assume, whenever this is needed, that n is sufficiently large.

We first prove two lemmas.

Lemma 3.1. *Let $G = (V, E)$ be a graph with at least $(9c - 15)(c + 1)n$ triangles and at most n vertices, then it contains a copy of F_c.*

Proof. Take a maximal set of edge-disjoint triangles in G, if they contain a subset of size at least c touching the same vertex then we are done. Otherwise, one can color these triangles with $3(c-2)+1 = 3c - 5$ colors so that no two triangles with the same color share a vertex (by simply coloring each triangle with the smallest available color). Each triangle in our original graph G shares an edge with one of these colored triangles, as they form a maximal set, so there is a set of unicolored triangles

5

with at least $\frac{(9c-15)(c+1)n}{3c-5} = 3(c+1)n$ triangles sharing edges with one of them (where here we are counting the colored triangles too).

Focusing on the triangles colored in this color and the ones sharing edges with them, note that there are at least $3(c+1)n$ of those organized in clusters, with each cluster consisting of one (colored) central triangle and all others sharing an edge with it. There are at most $n/3$ central triangles and hence more than $3cn$ triangles are not central, thus having two vertices in the center and one outside. Call the outside vertex the external one. There are $3cn$ of them, so there must be a vertex $v \in V$ which is an external vertex of at least $3c$ triangles. At most 3 triangles from each cluster can share an external vertex, so there are c triangles from different clusters sharing this vertex, and this is the only vertex they share. These c triangles form a copy of F_c, as needed. $\qquad\square$

Lemma 3.2. *For every $k > 3$ and n large enough there is a graph G on n vertices with at least $\Omega(n^{1+\frac{1}{k-1}})$ triangles and no cycles of length i for any i between 4 and k.*

We note that the exponent here can be slightly improved, at least for some values of k. In particular, for $k = 4$ the best possible value is $(1/6 + o(1))n^{3/2}$, as can be shown using the Erdős-Rényi graph [18], or Theorem 4.6 below with $t = 2$. For our purposes here, however, the above estimate suffices.

Proof. Let G' be a random graph on a fixed set of n labeled vertices obtained by choosing, randomly and independently, each of the $\binom{n}{3}$ potential triangles on the set of vertices to form a triangle in G' with probability $p = \frac{1}{2}n^{-\frac{2k-3}{k-1}}$. Let X be the random variable counting the number of triangles picked, and for $2 \leq i \leq k$ let Y_i denote the random variable counting the number of cycles of length i in which each edge comes from a different triangle. (In particular, Y_2 counts the number of pairs of selected triangles that share two vertices).

Note that if we remove one of our chosen triangles from each such cycle, then the resulting graph will contain no cycle of length between 4 and k. Indeed, if we have such a cycle using two edges of one triangle then replacing those by the third edge will create a shorter cycle, that cannot exist by assumption. Similarly, a cycle of length 4 cannot be created by two triangles if we leave no pair of triangles sharing two vertices. Put $Z = X - \sum_{i=2}^{k} Y_i$, and note that it is enough to show that the expectation of Z is at least $\Omega(n^{1+1/(k-1)})$. Indeed, if this is the case, then there is a graph G' for which the value of Z is at least $\Omega(n^{1+1/(k-1)})$. Fixing such a graph and omitting a triangle from each of the short cycles counted by the variables Y_i generates a graph G with the desired properties. Since $\mathbb{E}(X) = \binom{n}{3}p$ and

$$\mathbb{E}(Y_i) = \frac{n \cdot (n-1) \ldots (n-i+1)(n-2)^i}{2i}p^i \leq \frac{(n^2 p)^i}{2i} = \frac{n^{i/(k-1)}}{i2^{i+1}}$$

a simple computation shows that $\mathbb{E}(Z) \geq (1 + o(1))(1/12 - 1/128)n^{1+1/(k-1)}$, as needed. $\qquad\square$

We can now prove Theorem 1.1.

Proof of Theorem 1.1 . We start by showing that $ex(n, K_3, H)$ is linear in n for any extended friendship graph. Let H be an extended friendship graph with h vertices and let G be a graph on n vertices with at least $c(H)n$ triangles, where $c(H) = 10h^2$. We show that G must contain a copy of H.

We first show that G contains a subgraph with minimum degree at least h. As long as there is a vertex in G of degree smaller than h, omit it. This process must terminate with a nonempty graph containing more than $9h^2n$ triangles, since the total number of triangles that can be omitted this way is smaller than $\binom{h}{2}n < h^2n$. We can thus assume that the minimum degree in G is at least h, and that it has at most n vertices and at least $9h^2n$ triangles.

By Lemma 3.1 G contains a copy of the 2-core of H. This copy can be extended to a copy of H. Indeed, if H is disconnected add to it edges to make it connected (keeping the 2-core intact). We can now embed the missing vertices of H in G one by one, starting with the 2-core and always adding a vertex with exactly one neighbor in the previously embedded vertices. Since the minimum degree in G is at least h this can be done, providing the required copy of H.

To complete the proof of the theorem we have to show that if H is not a subgraph of an extended friendship graph then there is a graph G with n vertices and $\omega(n)$ triangles containing no copy of H. Note that H is not a subgraph of an extended friendship graph iff it either contains a cycle of length greater than 3 or it contains two vertex disjoint triangles. In the first case, Lemma 3.2 provides a graph G with a superlinear number of triangles containing no copy of H.

For the second case let G be the complete 3-partite graph $K_{1,\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$. Here all the triangles share a common vertex, hence no two are disjoint. As the number of triangles is $\lfloor \frac{(n-1)^2}{4} \rfloor$, this completes the proof. $\qquad\square$

Remark 3.1. *For any connected graph H with h vertices, an n vertex graph consisting of a disjoint union of $\lfloor n/(h-1) \rfloor$ cliques, each of size $h-1$, contains no copy of H and at least $\Omega(h^2n)$ triangles, showing that the estimate in the proof of the last theorem is tight, up to a constant factor.*

3.2 Cycles

In this subsection we prove Proposition 1.1, which (slightly) improves the estimates in [8] and [25]. We start with the proof of part (i). Let $G = (V, E)$ be a C_5-free graph on n vertices with the maximum possible number of triangles. Clearly we may assume that each edge of G lies in at least one triangle. Put $|E| = m$ and $\mathcal{N}(G, K_3) = t$. For each vertex $v \in V$ the graph spanned by its neighborhood $N(v)$ does not contain a path of length 3, and thus, by a known result of Erdős and Gallai [17], the number of edges it spans satisfies $|E(N(v))| \le d_v$, where $d_v = |N(v)|$ is the degree of v. The number of edges in $N(v)$ is exactly the number of triangles containing v and therefore

$$t \le \frac{\sum_v d_v}{3} = \frac{2m}{3} \tag{3}$$

Color the vertices of G randomly and independently, where each vertex is blue with probability p (which will be chosen later to be $p = 1/3$) and red with probability $1 - p$. For each edge $e = uv$ of G choose arbitrarily one vertex $w = w(e)$ such that u, v, w form a triangle. Denote by E' the set

of edges $e = uv$ of G so that both u and v are colored blue and w is colored red, and denote by V' the set of all blue vertices. Note that the graph (V', E') on the blue vertices contains no C_4 since otherwise each edge of this C_4 forms a triangle together with a red vertex, providing a copy of C_5 in G, which is impossible. Therefore

$$|E'| \leq ex(|V'|, C_4) = (\frac{1}{2} + o(1))|V'|^{\frac{3}{2}}.$$

Taking expectation in both sides and using linearity of expectation and the fact that the binomial random variable $|V'|$ is tightly concentrated around its mean we get

$$p^2(1-p)m \leq \mathbb{E}(|E'|) \leq (\frac{1}{2} + o(1))(np)^{\frac{3}{2}}.$$

This is because for each edge uv, the probability it belongs to E' is $p^2(1-p)$. Thus

$$m \leq (\frac{1}{2} + o(1))n^{\frac{3}{2}} \frac{1}{\sqrt{p(1-p)}}.$$

Since the right hand side is minimized when $p = \frac{1}{3}$ select this p to conclude that

$$m \leq (\frac{1}{2} + o(1))n^{\frac{3}{2}} \frac{3\sqrt{3}}{2}.$$

Plugging into (3) we get

$$t \leq (\frac{1}{2} + o(1))n^{\frac{3}{2}}\sqrt{3} = \frac{\sqrt{3}}{2}n^{\frac{3}{2}} + o(n^{\frac{3}{2}})$$

as needed. $\qquad\square$

The proof of part (ii) of Proposition 1.1 is similar. Here we do not optimize the value of the probability p and simply take $p = 1/2$, for small values of k the result can be slightly improved. To get the precise statement as stated in the proposition we use an additional trick. The details follow.

Let $G = (V, E)$ be a C_{2k+1}-free graph on n vertices with the maximum possible number of triangles. As before, assume that each edge of G lies in at least one triangle, and for each edge $e = uv$ of G choose a vertex $w = w(e)$ so that u, v, w form a triangle in G. Put $|E| = m$ and $\mathcal{N}(G, K_3) = t$. Since the neighborhood of any vertex v of G contains no path on $2k$ vertices, the Erdős-Gallai theorem implies that it contains at most $(k-1)d_v$ edges, implying that

$$t \leq \frac{\sum_v (k-1)d_v}{3} = \frac{2(k-1)m}{3} \qquad (4)$$

Split the vertices of G into $m = \lceil n/2 \rceil$ disjoint subsets, where if n is even each subset is of size 2 and otherwise one subset is of size 1. If a subset chosen is an edge uv of the graph G, we ensure that if $w = w(uv)$ then $u = w(vw)$ and $v = w(uw)$. As the subsets are disjoint, it is easy to check that such a choice is possible. Now color the vertices randomly red and blue: in each subset one vertex is colored red and the other is blue (where each of the two possibilities are equally likely). If n is odd then the vertex in the last class gets a random color. As before, let E' denote the set of edges $e = uv$ of G so that both u and v are colored blue and $w = w(e)$ is colored red, and denote by V' the set of

all blue vertices. The graph (V', E') contains no C_{2k} since otherwise we get a copy of C_{2k+1} in G, which is impossible. Thus

$$|E'| \leq ex(|V'|, C_{2k}) \leq ex(\lceil n/2 \rceil, C_{2k}) \tag{5}$$

since here $|V'|$ is always of cardinality either $\lceil n/2 \rceil$ or $\lfloor n/2 \rfloor$.

We claim that the expected cardinality of E' is at least $m/8$. Indeed, if for an edge uv with $w = w(uv)$ no pair of the three vertices u, v, w lie in a single subset, then the probability that u, v are blue and w is red is exactly $1/8$. For the other edges note that if uv forms one of our subsets and $w = w(uv)$, then the probability that uv lies in E' is 0, but the probability that uw lies in E' is $1/4$ and so is the probability that vw lies in E'. Hence the contribution from these three edges to the expectation of $|E'|$ is $2/4 > 3/8$. Linearity of expectation thus implies that the expected value of $|E'|$ is at least $m/8$ and thus by (5), $m/8 \leq ex(\lceil n/2 \rceil, C_{2k})$, and by (4)

$$t = \mathcal{N}(G, K_3) \leq \frac{16(k-1)}{3} ex(\lceil n/2 \rceil, C_{2k})$$

completing the proof. $\qquad \square$

Remark 3.2. *Bondy and Simonovits [11] proved that $ex(n, C_{2k}) \leq O(kn^{1+\frac{1}{k}})$. This has recently been improved by Bukh and Jiang [13] to $ex(n, C_{2k}) \leq O(\sqrt{k \log k}\, n^{1+\frac{1}{k}})$. Thus the upper bound obtained from the above proof is $ex(n, K_3, C_{2k+1}) \leq O(k^{3/2}\sqrt{\log k}\, n^{1+1/k})$.*

3.3 Books

An s-book is the graph consisting of s triangles, all sharing one edge.

Proposition 3.3. *For each fixed $s \geq 2$, if $H = H(s)$ is the s-book then $n^{2-o(1)} \leq ex(n, K_3, H) = o(n^2)$*

Proof. The lower bound follows from the construction of Ruzsa and Szemerédi [37], based on Behrend's construction [7] of dense subsets of the first n integers that contain no three term arithmetic progressions. This construction gives graphs on n vertices with

$$m = \frac{n^2}{e^{O(\sqrt{\log n})}} = n^{2-o(1)}$$

edges in which every edge is contained in a unique triangle. Therefore these graphs contain no 2-book, and hence no s-book, showing that

$$ex(n, K_3, H(s)) \geq m/3 \geq \frac{n^2}{e^{O(\sqrt{\log n})}} = n^{2-o(1)}.$$

The upper bound follows from the triangle removal lemma proved in [37]. If G is a graph on n vertices containing t triangles and no copy of $H(s)$, then every edge is contained in at most $s - 1$ triangles. Therefore, one has to remove at least $t/(s-1)$ edges of G in order to destroy all triangles. It follows that if $t \geq \epsilon n^2$ then, by the triangle removal lemma, the number of triangles in G is at least δn^3 for some $\delta = \delta(\epsilon, s) > 0$, and thus, by averaging, G contains an r-book for $r \geq 2\delta n > s$, contradiction. Thus $t = o(n^2)$, as needed. $\qquad \square$

4 Complete graphs and complete bipartite graphs

In this section we consider the cases in which T and H are either complete or complete bipartite graphs. Note that when both T and H are complete graphs the precise value of $ex(n, T, H)$ is known, as mentioned in Section 2. The following argument suffices to provide the precise value of $ex(n, K_{a,b}, K_t)$. If u and v are two non-adjacent vertices in a K_t-free graph G, then by making the set of neighbors of u identical to that of v (or vice versa), the graph stays K_t-free, and one can always choose one of these modifications to get a graph containing at least as many copies of $K_{a,b}$ as G. This is because every copy of $K_{a,b}$ in G that contains both u and v remains a copy in the modified graph as well. Repeating this procedure until every two nonadjacent vertices have the same neighborhoods we get a complete multipartite graph with n vertices and at most $t-1$ color classes, and one can now optimize the sizes of the color classes to obtain the maximum possible number of copies of $K_{a,b}$. (Note that this optimum is not necessarily obtained for equal or nearly equal color classes. Note also that the precise argument here requires to ensure the process above converges. To do so we can assign all potential copies of $K_{a,b}$ nearly equal algebraically independent weights, and always select the modification that maximizes the total weight of all copies obtained. It is not difficult to argue that for large n, in the extremal graph for any two distinct vertices, there are copies of $K_{a,b}$ containing exactly one of them, and therefore in the above process the total weight keeps increasing and it must converge.) The same argument shows that for any complete multipartite graph T with less than t color classes, the extremal graph giving the value of $ex(n, T, K_t)$ is itself a complete multipartite graph.

As to the case when $H = K_{s,t}$ for $s \le t$ and $T = K_m$, note that if $H = K_{1,t}$, then it is a star and avoiding it in a graph means bounding the degrees of the vertices. Thus to find $ex(n, K_m, K_{1,t})$ for $m \le t$ first note that as each vertex has degree at most $t-1$ the number of K_ms is at most $\frac{n}{m}\binom{t-1}{m-1}$. On the other hand, if n is divisible by t then taking $\frac{n}{t}$ disjoint copies of K_t will yield $\frac{n}{t}\binom{t}{m} = \frac{n}{m}\binom{t-1}{m-1}$ copies of K_m. If n is not divisible by t a similar bound can be achieved by taking $\lfloor \frac{n}{t} \rfloor$ copies of K_t and a clique on the remaining vertices. In [26] it is conjectured that the above construction is optimal, and this is proved for some specific cases.

For general m, s, t we start by proving Theorem 1.2. After that we prove another bound for cases that do not satisfy the assumptions of the theorem and then establish tighter results for several values of s, t when $T = K_3$.

To prove Theorem 1.2 in a more precise form we prove two lemmas, one for the upper bound and one for the lower.

Lemma 4.1. *For any fixed $m \ge 2$ and $t \ge s \ge m-1$*

$$ex(n, K_m, K_{s,t}) \le (\frac{1}{m!} + o(1))(t-1)^{\frac{m(m-1)}{2s}} n^{m - \frac{m(m-1)}{2s}}$$

Proof. We apply induction on m.

For $m = 2$, by the theorem of Kővari, Sós and Turán in [32]:

$$ex(n, K_2, K_{s,t}) = ex(n, K_{s,t}) \le (\frac{1}{2} + o(1))(t-1)^{\frac{1}{s}} n^{2 - \frac{1}{s}}.$$

10

This serves as our base case.

Now assume we have proved this for m and let us prove it for $m+1$. In what follows it will be convenient to use the means-inequality: For each $r < s$ and positive reals $x_1, \ldots, x_n$:

$$\sum_{i=1}^{n} x_i^r \leq n^{1-r/s} (\sum_{i=1}^{n} x_i^s)^{r/s}.$$

Let $G = (V, E)$ be a $K_{s,t}$ free graph on n vertices, and let us bound the number of copies of K_{m+1} in it. For each $v \in V$ we know that its neighborhood $N(v)$ does not contain any copy of $K_{s-1,t}$. By the induction assumption we can bound the number of copies of K_m in $N(v)$:

$$\mathcal{N}(N(v), K_m) \leq ex(d_v, K_m, K_{s-1,t}) \leq (\frac{1}{m!} + o(1))(t-1)^{\frac{m(m-1)}{2(s-1)}} d_v^{m - \frac{m(m-1)}{2(s-1)}}$$

By bounding the number of K_m in each $N(v)$ we can bound the number of K_{m+1} in G resulting in:

$$\mathcal{N}(G, K_{m+1}) \leq \frac{1}{m+1} (\frac{1}{m!} + o(1))(t-1)^{\frac{m(m-1)}{2(s-1)}} \sum_v d_v^{m - \frac{m(m-1)}{2(s-1)}}$$

$$\leq (\frac{1}{(m+1)!} + o(1))(t-1)^{\frac{m(m-1)}{2(s-1)}} (\sum_v d_v^s)^{\frac{m(2s-m-1)}{2s(s-1)}} n^{1 - \frac{m(2s-m-1)}{2s(s-1)}} \tag{6}$$

$$\leq (\frac{1}{(m+1)!} + o(1))(t-1)^{\frac{(m+1)m}{2s}} n^{\frac{m(2s-m-1)}{2(s-1)} + 1 - \frac{m(2s-m-1)}{2s(s-1)}} \tag{7}$$

$$= (\frac{1}{(m+1)!} + o(1))(t-1)^{\frac{(m+1)m}{2s}} n^{(m+1) - \frac{(m+1)m}{2s}}$$

Here we used the means inequality to get the first inequality (an easy calculation shows that $m - \frac{m(m-1)}{2(s-1)} < s$). To bound the sum $\sum_v d_v^s$ we used the fact that the number of s-edged stars in G cannot exceed $\binom{n}{s}(t-1)$ because otherwise t of them will share the same s leaves, creating a $K_{s,t}$. $\square$

Lemma 4.2. *For any fixed m, $s \geq 2m - 2$ and $t \geq (s-1)! + 1$*

$$ex(n, K_m, K_{s,t}) \geq (\frac{1}{m!} + o(1))n^{m - \frac{m(m-1)}{2s}}$$

Proof. We use the projective norm-graphs constructed in [4], where it is shown that $H(q,s)$ has $n = (1 + o(1))q^s$ vertices, is $d = (1 + o(1))q^{s-1}$-regular, and is $K_{s,(s-1)!+1}$ free. An (n, d, λ) graph is a d-regular graph on n vertices in which all eigenvalues but the first have absolute value at most λ. As shown in [3] (see also [31], Theorem 4.10) the following holds: Let G_1 be a fixed graph with r edges, s vertices and maximum degree Δ. Let G_2 be an (n, d, λ) graph. If $n \gg \lambda(\frac{n}{d})^\Delta$ then the number of copies of G_1 in G_2 is $(1 + o(1))\frac{n^s}{|Aut(G_1)|}(\frac{d}{n})^r$.

In our case we take $G_1 = K_m$ and $G_2 = H(q, s)$. By the results in [41] or [5] we know that the second eigenvalue, in absolute value, of $H(q, s)$ is $q^{\frac{s-1}{2}}$, thus to get the inequality $n \gg \lambda(\frac{n}{d})^\Delta$ it suffices that $m < \frac{s+3}{2}$. Plugging the choice of G_1, G_2 into the result mentioned above implies:

$$\mathcal{N}(H(q,s), K_m) = (1 + o(1))\frac{n^m}{m!}(\frac{1}{q})^{\binom{m}{2}}$$
$$= (\frac{1}{m!} + o(1))(q^s - q^{s-1})^m (\frac{1}{q})^{\binom{m}{2}}$$
$$= (\frac{1}{m!} + o(1))q^{s(m - \frac{m(m-1)}{2s})}$$
$$= (\frac{1}{m!} + o(1))n^{m - \frac{m(m-1)}{2s}}$$

$\square$

Note that for $m = 3$ the lower bound above applies only for $s \geq 4$. The following result provides a similar bound for $s \in 2, 3$ as well.

Lemma 4.3. *For any fixed $s \geq 2$ and $t \geq (s-1)! + 1$ we have $ex(n, K_3, K_{s,t}) = \Theta(n^{3 - \frac{3}{s}})$*

Proof. In view of the previous upper bound it suffices to show the existence of a graph G with n vertices containing no copy of $K_{s,t}$ and containing at least $\Omega(n^{3 - \frac{3}{s}})$ triangles. For this we apply again the projective norm-graphs $H(q,s)$ constructed in [4], which are $K_{s,t}$ free.

The graph $H = H(q,s)$ is defined in the following way: $V(H) = GF(q^{s-1}) \times GF(q)^*$ where $GF(q)^*$ is the multiplicative group of the q element field. For $A \in GF(q^{s-1})$ define the norm

$$N(A) = A \cdot A^q \ldots A^{q^{s-2}}.$$

Two vertices (A, a) and (B, b) are connected in H if $N(A + B) = ab$. Note that $|V(H)| = q^s - q^{s-1}$ and H is $q^{s-1} - 1$ regular.

We need to show that $H(q,s)$ has the right number of triangles. As mentioned above, the eigenvalues and multiplicities of $H(q,s)$ are given in [41], [5]. These are as follows: $q^{s-1} - 1$ is of multiplicity 1, 0 is of multiplicity $q - 2$, 1 and -1 are of multiplicity $(q^{s-1} - 1)/2$ each, and $q^{(s-1)/2}, -q^{(s-1)/2}$ are of multiplicity $(q^{s-1} - 1)(q-2)/2$ each. Summing the cubes of the eigenvalues we conclude that the number of closed walks of length 3 in $H(q,s)$ is $(q^{s-1} - 1)^3 = (1 + o(1))q^{3s-3}$.

A closed walk of length 3 is not a triangle iff it contains a loop. Fixing $A \in GF(q^t)$ the vertex (A, x) has a loop iff $N(A + A) = x^2$. There are at most 2 solution x for each given A. Thus there are no more than $2q^{s-1}$ loops. A closed walk of length 3 containing a loop must also contain an additional edge taken twice (this additional edge may also be the loop itself). As the graph is $q^{s-1} - 1$ regular we get at most $6q^{s-1}q^{s-1} = o(q^{3s-3})$ such walks containing a loop. As the number of closed walks of length 3 is $(1 + o(1))q^{3s-3}$ this is negligible and the number of triangles is $(\frac{1}{6} + o(1))q^{3s-3} = \Theta(|V(H)|^{3 - 3/s})$, as needed. $\square$

Remark 4.1. *For the special case of $s = t = 3$ it can be shown that the construction of Brown [12] gives another example of a $K_{3,3}$-free on n vertices with essentially the same number of triangles.*

Remark 4.2. *The number of triangles in the projective norm graphs is also computed in a recent paper of Kostochka, Mubayi and Verstraëte [30], motivated by an extremal problem for 3-uniform hypergraphs. They estimate this number directly, without using the eigenvalues.*

For values of s, t and m that do not satisfy the restrictions in the previous results we provide slightly weaker results in the following lemmas:

Lemma 4.4. *For any fixed m and $t \geq s \geq 1$ such that $t + s > m$*

$$ex(n, K_m, K_{s,t}) \leq (1 + o(1)) \frac{(m - s)!(t - 1)^{\frac{s-1}{2}}}{m!} \binom{t - 1}{m - s} n^{\frac{s+1}{2}}$$

Proof. We apply induction on s. As the base case take $s = 1$. In this case the fact that G is $K_{1,t}$ free implies that the degrees of all vertices are at most $t - 1$. Thus each vertex can take part in no more than $\binom{t-1}{m-1}$ copies of K_m and hence

$$ex(n, K_m, K_{1,t}) \leq \frac{1}{m} \binom{t - 1}{m - 1} n$$

Note that if $t \mid n$ then this bound is achieved by the disjoint union of $\frac{n}{t}$ pairwise vertex disjoint copies of K_t.

Assuming the result for $s - 1$ we prove it for s. If G is $K_{s,t}$ free, then for each $v \in V$ its neighborhood cannot contain a copy of $K_{s-1,t}$. By the induction hypothesis this bounds the number of copies of K_{m-1} by

$$(1 + o(1)) \frac{(m - s)!(t - 1)^{\frac{s-2}{2}}}{(m - 1)!} \binom{t - 1}{m - s} d_v^{\frac{s}{2}}$$

where d_v is the degree of v. This is clearly also the number of copies of K_m containing v. Therefore,

$$\mathcal{N}(G, K_m) \leq \frac{1}{m} (1 + o(1)) \sum_v \frac{(m - s)!(t - 1)^{\frac{s-2}{2}}}{(m - 1)!} \binom{t - 1}{m - s} d_v^{\frac{s}{2}}$$

$$\leq (1 + o(1)) \frac{(m - s)!(t - 1)^{\frac{s-2}{2}}}{m!} \binom{t - 1}{m - s} \left(\sum_v d_v^s \right)^{\frac{1}{2}} n^{\frac{1}{2}} \tag{8}$$

$$\leq (1 + o(1)) \frac{(m - s)!(t - 1)^{\frac{s-1}{2}}}{m!} \binom{t - 1}{m - s} n^{\frac{s+1}{2}} \tag{9}$$

where we get (8) from the means inequality and (9) from the fact that we cannot have more than $\binom{n}{s}(t - 1)$ copies of s stars in G. $\qquad \square$

Note that unlike Theorem 1.2 to get the bound in Lemma 4.4 we need to assume nothing but the obvious fact that K_m does not contain a copy of $K_{s,t}$. On the other hand for every $m, s \in \mathbb{N}$ one has $\frac{s+1}{2} \geq m - \frac{m(m-1)}{2s}$ where we have an equality when $s = m - 1$ and $s = m$. Thus when $s < m - 1$ we must use Lemma 4.4, but if $s \geq m - 1$ Lemma 4.1 gives a better upper bound.

Lemma 4.5. *For any m and $t > m - 2 > 1$*

$$ex(n, K_m, K_{2,t}) \geq \frac{1}{4} m^{\frac{-4m}{3}} n^{\frac{4}{3}}$$

Proof. In [33] Lazebnik and Verstraëte show that there exists an $m-$uniform hypergraph H on n vertices, with at least $\frac{1}{4}m^{\frac{-4m}{3}}n^{\frac{4}{3}}$ hyperedges and with girth at least 5. Let G be the graph obtained from H by replacing each hyperedge of H by a copy of K_m. We next observe that G contains no copy of $K_{2,t}$.

Assume towards a contradiction that G contains a copy of $K_{2,t}$. As $t > m - 2$ the copy of $K_{2,t}$ cannot be contained in a single K_m and so there must be at least two edges in it that come from two different K_ms. These two edges are a part of a C_4 hence in the hypergraph H this C_4 has vertices in at least two hyperedges. Thus H must contain a cycle of length between 2 and 4 in contradiction to the assumption that H has girth at least 5. Therefore G is $K_{2,t}$ free with at least $\frac{1}{4}m^{\frac{-4m}{3}}n^{\frac{4}{3}}$ copies of K_m, as needed. $\qquad\square$

Finally for $s = 2$ we can determine the asymptotic behavior of $ex(n, K_3, K_{2,t})$ up to a lower order term, as shown next.

Theorem 4.6. *For any fixed* $t \geq 2$, $ex(n, K_3, K_{2,t}) = (1 + o(1))\frac{1}{6}(t - 1)^{3/2}n^{3/2}$.

Proof. The upper bound follows from the assertion of Lemma 4.1 with $m = 3$ and $s = 2$. To prove the lower bound we apply a construction of Füredi [22], extending the one of Erdős and Rényi [18]. The details follow. Let $\mathbb{F}$ be a finite field of order q, where $t - 1$ divides $q - 1$, and let h be a nonzero element of $\mathbb{F}$ that generates a multiplicative subgroup $A = \{h, h^2, ..., h^{t-1} = 1\}$ of order $t - 1$ in $\mathbb{F}^*$. The vertices of the graph $G = G(\mathbb{F}, t - 1)$ are all nonzero pairs in $(\mathbb{F} \times \mathbb{F})$, where two pairs (a, b) and (a', b') are considered equivalent if for some $h^\alpha \in A$, $h^\alpha a = a'$ and $h^\alpha b = b'$. Two vertices $(a, b), (c, d)$ are connected if $ac + bd \in A$. The number of vertices of G is $n = (q^2 - 1)/(t - 1)$ and it is not difficult to check that it is regular of degree q, where here each loop adds one to the degree. Indeed, for a fixed vertex (b, c) and for each $h^\alpha \in A$ there are exactly q solutions (x, y) to the equation $bx + cy = h^\alpha$, and as any neighbor (x, y) of (b, c) is obtained this way $t - 1$ times, by our equivalence relation, the graph is q-regular. Note that there is a (unique) loop at a vertex (x, y) iff $x^2 + y^2 \in A$. For each fixed $h^\alpha \in A$ and each fixed $x \in \mathbb{F}$ there are at most 2 solutions for y, showing that the number of loops is at most $2q(t - 1)/(t - 1) = 2q$ (it is in fact smaller, but this estimate suffices for us).

It thus follows that the number of edges of G (without the loops) is $m = (\frac{1}{2} + o(1))q^3/(t - 1) = (\frac{1}{2} + o(1))\sqrt{t - 1}\, n^{3/2}$.

We claim that any two distinct vertices (a, b) and (c, d) of G have exactly $t - 1$ common neighbors (if there is a loop in one of these vertices and they are adjacent, this counts as a common neighbor). Indeed, the vertex (x, y) is a common neighbor if for some $0 \leq \alpha, \beta \leq t - 2$

$$ax + by = h^\alpha$$
$$cx + dy = h^\beta.$$

These two equations are linearly independent, and hence there is a unique solution for each choice of α, β. As the number of choices for α and β is $(t - 1)^2$, and every common neighbor is counted this way $t - 1$ times, the claim follows.

By the above claim, G is $K_{2,t}$-free. In addition, since the endpoints of each edge in it have $t - 1$ common neighbors, each edge is contained in $t - 1$ triangles (including the degenerated ones containing a loop). The number of triangles containing a loop is smaller than $2q^2$ which is far smaller than the number of edges $m = \Theta(q^3/(t - 1))$. Therefore, the number of triangles is

$$(1 + o(1))\frac{1}{3}m(t - 1) = (1 + o(1))\frac{1}{6}(t - 1)\sqrt{t - 1}\, n^{3/2}$$

completing the proof. □

We conclude the section by considering the case $T = K_{a,b}$ and $H = K_{s,t}$ where we establish the following.

Proposition 4.7. *(i) If $a \leq s \leq t$ and $a \leq b < t$ then*

$$ex(n, K_{a,b}, K_{s,t}) \leq (1 + o(1))\frac{1}{a!(b!)^{1-a/s}}\left(\frac{t - 1}{b}\right)^{a/s} n^{a+b-ab/s},$$

and if $a = b$ the above bound can be divided by 2.
(ii) If $(a - 1)! + 1 \leq b < (s + 1)/2$ then for all $t \geq s$, $ex(n, K_{a,b}, K_{s,t}) = \Theta(n^{a+b-ab/s})$.

Proof. (i) Let $G = (V, E)$ be a $K_{s,t}$-free graph on n vertices. For each subset B of b vertices, let n_B denote the number of common neighbors of all vertices in B. The number of copies of $K_{a,b}$ in G is clearly exactly $\sum_B \binom{n_B}{a}$ for $b < a$, where the summation here and in what follows is over all b-subsets B of V. If $a = b$ the right hand side should be divided by 2. We proceed with the case $a < b$ recalling that a factor of $1/2$ can be added if $a = b$. By the means inequality, the number of copies of $K_{a,b}$ in G is at most

$$\frac{1}{a!}\sum_B n_B^a \leq \frac{1}{a!}\binom{n}{b}^{1-a/s}\left(\sum_B n_B^s\right)^{a/s}$$

$$\leq (1 + o(1))\frac{n^{b-ab/s}}{a!(b!)^{1-a/s}}\left(\left(\frac{t - 1}{b}\right)n^s\right)^{a/s} = (1 + o(1))\frac{1}{a!(b!)^{1-a/s}}\left(\left(\frac{t - 1}{b}\right)\right)^{a/s}n^{a+b-ab/s}.$$

Here we used the fact that $\sum_B n_B^s \leq (1 + o(1))\binom{t-1}{b}n^s$ since if we have more than $\binom{t-1}{b}$ subsets of cardinality b in V with each of them having the same s-subset among their common neighbors, then we get a copy of $K_{s,t}$, which is impossible.

(ii) The projective norm graphs give, as in the proof of Lemma 4.2, that if $(a-1)!+1 \leq b < (s+1)/2$ then $ex(n, K_{a,b}, K_{s,t}) \geq \Omega(n^{a+b-ab/s})$. This and part (i) supply the assertion of part (ii). □

5   Forbidding a fixed tree

In this section we prove Theorems 1.5 and 1.7.

5.1 Proof of Theorem 1.5

Let $G = (V, E)$ be a graph, and let T be a tree on a set $V(T) = \{u_1, \ldots, u_t\}$ of t vertices. Let $V = V_1 \cup V_2 \cup \cdots \cup V_t$ be a partition of V into t pairwise disjoint sets. Call a copy of T in G *proper* if its vertices are $v_1, v_2, \ldots, v_t$ where $v_i \in V_i$ and v_i plays the role of u_i in this copy. For a subset $U \subset V(T)$ and an integer h, call a (U, h)-blow-up of T *proper* if each vertex v of the blow-up belongs to V_i if and only if it plays the role of u_i in the blow up.

Recall that a graph is h'-degenerate if every subgraph of it contains a vertex of degree at most h'. It is easy and well known that if a graph is not h'-degenerate then it contains a copy of any tree with h' edges.

The main part of the proof is the following lemma.

Lemma 5.1. *For every positive integers t, h, h', m there is a $C(t, h, h', m)$ so that the following holds. Let T be a tree on a set $V(T)$ of t vertices, let $G = (V, E)$ be a graph on n vertices and let $V = V_1 \cup \cdots \cup V_t$ be a partition of V. If G is h' degenerate and it contains at least $C(t, h, h', m)n^{m-1}$ proper copies of T, then it must contain a proper (U, h) blow-up of T, such that $T \backslash U$ has m connected components.*

Proof. We apply induction on $m + t$.

The base case m+t=3: In this case $t = 2$, that is, T is an edge, and $m = 1$. Let $G = (V, E)$ and $V = V_1 \cup V_2$ be a graph and a partition of its vertex set as in the statement of the lemma, and suppose it contains at least $(h-1)^2 + 1$ proper copies of T. These copies form a bipartite graph with vertex classes V_1, V_2 and hence, by König's Theorem, it must contain either a star or a matching of size h. A matching is a proper $(\emptyset, h)$ blow-up of $T = K_2$ and a star is a proper $(\{v\}, h)$ blow-up, where v is one of the vertices of T. In both cases $T \backslash U$ has 1 connected component. This establishes the base case.

Induction step Assuming the assertion holds for any m and t satisfying $m + t < k$, we prove it for $m + t = k$, $(k \geq 4)$. Let T be a tree on t vertices, and suppose that $V(T) = \{u_1, ..., u_t\}$ where u_1 is a leaf and u_2 is its unique neighbor. Let $G = (V, E)$ be an h'-degenerate graph with n vertices, and let $V = V_1 \cup \ldots V_t$ be a partition of its vertex set. Suppose that G contains at least $C(t, h, h', m)n^{m-1}$ proper copies of T. We have to show that it contains a proper (U, h) blow-up of T such that $T \backslash U$ has m connected components.

For each vertex $v \in V_2$ let $d_1(v)$ be the number of its neighbors in V_1. Furthermore, put $T' = T \backslash \{u_1\}$ and let $\mathcal{N}_{u_2}(T', v)$ be the number of copies of T' in which v plays the role of u_2 and for each $2 < i \leq t$ the vertex playing the role of u_i lies in V_i. The following clearly holds, with $C = C(t, h, h', m)$:

$$Cn^{m-1} \leq \sum_{v \in V_2} d_1(v) \cdot \mathcal{N}_{u_2}(T', v)$$

$$= \sum_{v \in V_2, d_1(v) \geq h} d_1(v) \cdot \mathcal{N}_{u_2}(T', v) + \sum_{v \in V_2, d_1(v) < h} d_1(v) \cdot \mathcal{N}_{u_2}(T', v)$$

One of the summands must be at least $\frac{C}{2}n^{m-1}$. We consider both cases.

<u>Case 1</u>: $\sum_{v \in V_2, d_1(v) \geq h} d_1(v) \cdot \mathcal{N}_{u_2}(T', v) \geq \frac{C}{2} n^{m-1}$.

If $m = 1$, then there is a vertex $v \in V_2$ with $d_1(v) \geq h$ and $\mathcal{N}_{u_2}(T', v) \geq 1$. This implies the existence of a proper $(V(T) \setminus \{u_1\}, h)$ blow-up of T. If $m > 1$, as G has at most $h'n$ edges (since it is h'-degenerate), $\sum_{v \in V_2, d_1(v) \geq h} d_1(v) \leq |E(G)| \leq h' \cdot n$, and thus there must be a vertex v_2 for which $\mathcal{N}_{u_2}(T', v_2) \geq \frac{C}{2h'} n^{m-2}$.

Consider the induced subgraph G' of G on the set of vertices $V' = \{v_2\} \cup V_3 \cup \cdots \cup V_t$, with this partition into $(t-1)$ disjoint sets. This graph contains all the proper copies of T' in which v_2 plays the role of u_2. G' is also h'-degenerate, and contains at least $\frac{C'}{2h} n^{m-2}$ proper copies of T'. As $|V(T')| = t - 1$ (and also $m - 1 < m$) we can use the induction assumption on G', and find a proper (U, h) blow up of T' in G' in which $T' \setminus U$ has $m - 1$ connected components and $u_2 \in U$ as there is only one vertex, $v_2 \in V_2$, and hence only v_2 can play the role of u_2 in G'. The same set U thus gives the required proper (U, h) blow up of T, as $T \setminus U$ has all of the connected components of $T' \setminus U'$ together with a new connected component which is $\{u_1\}$. There is a copy of this proper (U, h) blow up of T as we can complete the (U, h) blow up of T' with h neighbors of v_2.

<u>Case 2</u>: $\sum_{v \in V_2, d_1(v) < h} d_1(v) \cdot \mathcal{N}_{u_2}(T', v) \geq \frac{C}{2} n^{m-1}$.

Let G' be the induced subgraph of G on $V_2' \cup \cdots \cup V_t$, where V_2' is the set of all vertices of V_2 satisfying $1 \leq d_1(v) < h$. As $\sum_{v \in V_2, 1 \leq d_1(v) < h} d_1(v) \mathcal{N}_{u_2}(T', v) \geq \frac{C}{2} n^{m-1}$ and $|V(T')| = t - 1$ we can use the induction assumption and find a proper (U', h^2) blow-up of T' with $T' \setminus U'$ having m connected components. It is left to complete this blow-up to the required proper (U, h) blow up of T.

Consider the vertices that play the role of u_2 in the (U', h^2) blow up of T'. There are two options: either there is only one such vertex or there are h^2 of them. If it is a single vertex then we complete the blow-up into a proper (U, h) blow up of T by taking $U = U' \cup \{u_1\}$ (this is actually a (U, h^2) blow-up). If there are h^2 of them, consider the bipartite graph consisting of h^2 edges, 1 from each copy of u_2 to an arbitrarily chosen neighbor of it in V_1. This graph must contain either a matching of size h or a star with h edges. A matching will leave us with the same $U = U'$ and for a star we take $U = U' \cup \{u_1\}$. In both cases the proper (U, h) blow up is contained in G and $T \setminus U$ has m connected components. $\qquad \square$

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Let H and T be trees with h and t vertices, respectively, and suppose $m = m(T, H)$. By the definition of $m(T, H)$ there exists a U such that $T \setminus U$ has m connected components, and the (U, h) blow-up of T has no copy of H. Using the same set U define G_T to be a $(U, \frac{n - |U|}{t - |U|})$ blow-up of T.

We next show that G_T is H free and has at least $c_1(t, h) n^m$ copies of T. If there is a copy of H in G_T, then it uses h vertices and is thus contained in a (U, h) blow up of T. But by the definition of m this blow-up is H free, so G_T must be H free too. To find $c_1(t, h) n^m$ copies of T, recall that $T \setminus U$ has m connected components, and note that any choice of a copy of each connected component can be completed into a copy of T.

On the other hand, if $ex(n, T, H) > t^t C(t, h, h, m) n^m$ take a random partition of the vertex set V of the extremal graph G into t pairwise disjoint sets $V_1, \ldots, V_t$. Each fixed copy of T becomes a proper copy with respect to this partition with probability $1/t^t$. Thus, by linearity of expectation, the expected number of proper copies is at least $C(t, h, h, m) n^m$, and hence there is a partition with at least that many proper copies. As G is H-free it is h-degenerate. By the lemma above it contains a (U, h) blow-up of T where $T \setminus U$ has $m+1$ connected components. This contradicts the maximality of $m(T, H)$, so $ex(n, T, H) \leq t^t C(t, h, h, m) n^m$. Finally define $c_2(t, h) = \max_{1 \leq m \leq t/2} C(t, h, h, m)$. $\qquad \square$

5.2 Proof of Theorem 1.7

We need the following well known result, see, for example, Theorem 7.3 in [10] for a proof.

Lemma 5.2. *In any bipartite graph with no isolated vertices, the number of vertices in a maximum independent set is equal to the number of edges in a minimum edge-cover.*

Proof of Theorem 1.7 . We prove the equivalence of the statements by deriving (2) from (1), (3) from (2) and (1) from (3).

$1 \Rightarrow 2$ Assume $ex(n, T, H) = \Theta(n^{\alpha(T)})$. Fix any minimum edge cover Γ of T and enumerate the vertices of T arbitrarily $\{u_1, .., u_t\}$, where t is the number of vertices of T . Let $G = G_T$ be an extremal graph, that is, a graph on n vertices with $cn^{\alpha(T)}$ copies of T and no copy of H. Enumerate the vertices of G_T randomly, and call a copy of T monotone if it is spanned by a set of vertices enumerated $i_1 < .. < i_t$ where i_j plays the role of u_j in T. By linearity of expectation the expected number of monotone copies of T in G is $\frac{c}{t!} n^{\alpha(T)} = c' n^{\alpha(T)}$. Fix a numbering with at least that many monotone copies. We next show that this graph must contain a (U, h) blow up of T for some choice of a $U(\Gamma)$-set U.

Denote the set of edges in the edge cover Γ by $\{e_1, .., e_{\alpha(T)}\}$. We can map the monotone copies of T to choices of edges that play the role of Γ so there must be at least $c' n^{\alpha(T)} = c' n^{|\Gamma|}$ such choices (the equality is by Lemma 5.2). Consider the following hypergraph $\mathcal{H} = (V_{\mathcal{H}}, E_{\mathcal{H}})$. The vertices $V_{\mathcal{H}}$ are the edges of G and a set $\{v_1, ..., v_{\alpha(T)}\}$ forms an edge in $E_{\mathcal{H}}$ if the corresponding edges in G span a monotone copy of T, where each edge plays its enumerated role in Γ. By the assumption on G, we have $|E_{\mathcal{H}}| = c' n^{\alpha(T)}$, and $|V_{\mathcal{H}}| \leq hn$. By the main theorem in [15] there is a $K_{s,..,s}^{\alpha(T)}$ in our hypergraph, where $s = h^2$, provided n is sufficiently large. Therefore there are disjoint sets of vertices $U_1, .., U_{\alpha(T)} \subset V_{\mathcal{H}}$ such that for any choice of $u_i \in U_i$, $\{u_1, ..., u_{\alpha(T)}\} \in E_{\mathcal{H}}$ and $|U_i| = h^2$ for all i.

Returning to G_T note that the $K_{s,..,s}^{\alpha(T)}$ in our hypergraph provides pairwise disjoint sets of edges $E_1, .., E_{\alpha(T)} \subset E(G_T)$, each of size $s = h^2$, such that any choice of a single edge from each set E_i spans a monotone copy of T in G, where the edge from E_i plays the role of the edge e_i in the copy. We next show that $E_1, .., E_{\alpha(T)}$ and the edges connecting them in G contain a (U, h) blow up of T, with U being a $U(\Gamma)$-set.

To this end we define the set $U \subset V(T)$. First note that a minimum edge cover does not contain a path of length 3 and hence Γ must be a union of stars and single edges. Define the complement U^c of U in the following way. If the edges $e_{i_1}, .., e_{i_k} \in E(T)$ form a star in Γ take the leaves of the star

into U^c. For a single edge in Γ, say e_j, consider the corresponding set E_j. It must contain either a star with at least h edges or a matching with at least h edges. If it contains an h-star, take into U^c the endpoint which is not the center of the star. If it contains a matching of size h, take into U^c both the vertices of e_j. Finally, put $U = V(T) \setminus U^c$.

It is left to show that U is indeed a $U(\Gamma)$-set with the required properties. We first show that the connected components in $T \setminus U$ have the needed properties. As $V(T) \setminus U = U^c$, the choice of U^c ensures that each edge in Γ has a non-empty intersection with a connected component in $T \setminus U$. It is left to show that no connected component in U^c can intersect two edges of Γ.

Assume towards contradiction that there is a connected component in U^c that intersects more than one edges of Γ. Then there must be two vertices u_i, u_j chosen into U^c that are connected by an edge of $E(T) \setminus \Gamma$. This means that in G_T there is a set of vertices of size at least h corresponding to u_j and another set of size h corresponding to u_i, where any choice of two vertices, one from each set, can be completed into a copy of T. Thus all the vertices in the first set must be connected to all those in the second. But in this case G_T contains a complete bipartite graph $K_{h,h}$ and hence contains a copy of H, contradicting the assumptions.

We conclude that in $T \setminus U$, each connected component intersects a single edge of Γ, and that each edge in Γ intersects a single connected component in $T \setminus U$, as needed. Note also that by the discussion above $T \setminus U$ is a vertex disjoint union of edges and single vertices.

G_T contains a (U, h) blow-up of T, as it contains h copies of each vertex in U^c and they are connected in G as needed to form the required blow-up.

$2 \Rightarrow 3$ This is obvious.

$3 \Rightarrow 1$ Assume there is a minimum edge cover Γ of T and a (U, h) blow up of T that does not contain a copy of H with U being a $U(\Gamma)$ set. Any H free graph has at most hn edges, thus by [1] the number of copies of T in such a graph is at most $O(n^{\alpha(T)})$, providing the required upper bound.

For the lower bound let G_T be a $(U, \frac{n-|U|}{t-|U|})$ blow up of T. We claim that if the (U, h) blow up of T does not contain a copy of H, then G_T does not contain one either. Indeed, as in the proof of Theorem 1.5, if we assume that the $(U, \frac{n-|U|}{t-|U|})$ blow-up does contain a copy of H then this copy uses at most h vertices and hence must be contained in the (U, h) blow-up as well, contradicting the assumption.

$T \setminus U$ has $\alpha(T) = |\Gamma|$ connected components as U is a $U(\Gamma)$-set. There are $\Theta(n)$ choices for each connected component in $T \setminus U$, and each choice produces a copy of T in G_T. Thus we have $\Theta(n^{\alpha(T)})$ copies and $ex(n, T, H) = \Theta(n^{\alpha(T)})$. $\qquad \square$

Theorem 1.7 provides a characterization of the pairs (T, H) of a bipartite graph T and a tree H for which $ex(n, T, H) = \Theta(n^{\alpha(T)})$. This characterization yields the following result about the complexity of the corresponding algorithmic problem.

Theorem 5.1. *The problem of deciding, for a given input consisting of a bipartite graph T and a tree H, if $ex(n, T, H) = \Theta(n^{\alpha(T)})$ is $co-NP$-hard.*

Proof. Let G be a graph on m vertices with minimum degree at least 4. Let T be the bipartite graph $(A \cup B, E)$ with A being the set of vertices of G, $V(G)$, and B being the set of edges of G, $E(G)$.

The edges of the bipartite graph are as follows, a couple $\{v, e\}$ is an edge in T if $v \in e$ in G. Let H be a path of length $2m + 1$.

Claim 5.2. $ex(n, T, H) = \Theta(n^{\alpha(T)})$ *if and only if the graph G does not contain a Hamilton path.*

Before proving this claim note that it is well known that the problem of deciding if an input graph G contains a Hamilton path is NP-complete (see, e.g., [24]). It is not difficult to show that for any fixed δ (and in particular for $\delta = 4$) this problem remains NP-hard even when restricted to input graphs of minimum degree at least δ. To see this, consider a graph F with minimum degree d. Let F' be the graph obtained from F by adding to it a set of $(d + 2)$ new vertices that form a clique, and by joining one of the vertices of this clique to all vertices of G. It is easy to check that F has a Hamilton path if and only if F' has such a path, and that the minimum degree of F' is $d + 1$. Repeating this argument we conclude that indeed the Hamilton path problem remains NP-hard when restricted to input graphs of minimum degree at least δ. It thus suffices to prove Claim 5.2 in order to establish the theorem.

To prove this claim, we choose a specific minimum edge cover Γ in T, and show that for this edge cover checking if the second condition in Theorem 1.7 holds is equivalent to deciding if the graph G contains a Hamilton path.

Since the degree of each vertex in A is at least 4 and the degree of each vertex of B is 2, it follows from Hall's Theorem that the graph T contains $|A|$ vertex disjoint stars, one centered at each vertex of A, with each star having two leaves. We can now complete these arbitrarily into a minimum edge cover, by connecting each vertex of B that does not lie in these stars to an arbitrary neighbor in A. This provides a minimum edge cover Γ in which every connected component is a star with at least two leaves.

The only possible $U(\Gamma)$-set for this Γ is A. The $(A, 2m + 2)$ blow-up of T contains a copy of H, which is a path of length $2m + 1$, if and only if G contains a Hamilton path, as such a path must alternate between A and B, and can visit each vertex in A only once.

$\square$

6 Concluding remarks and open problems

- In Section 5 we have shown several cases in which when H is a tree, $ex(n, T, H) = \Theta(n^k)$ where k is an integer. We believe that this phenomenon is more general, and that if H is a tree then for any graph T, $ex(n, T, H) = \Theta(n^{k(T,H)})$ for some integer $k = k(T, H)$.

- One of the cases we focused on is $ex(n, K_3, H)$. Even in this special case there are many difficult problems that remain open. One such problem that received a considerable amount of attention is the case that H is the 2-book, that is, two triangles sharing an edge. This is equivalent to the problem of obtaining tight bounds for the triangle removal lemma, which is wide open despite the fact we know that here $n^{2-o(1)} \leq ex(n, K_3, H) \leq o(n^2)$ and despite some recent progress in [20].

The determination of $ex(n, K_3, H)$ is complicated in many cases, and we do not even know its correct order of magnitude for some simple graphs like odd cycles. In this specific case, however, it may be that the lower bound in (2) and the upper bound in Proposition 1.1 differ only by a constant factor, as it may be true that the functions $ex(m, C_{2k})$ and $ex_{bip}(m, C_4, C_6, \ldots, C_{2k})$ differ only by a constant factor. The problem of determining the correct order of magnitude of $ex(n, K_3, K_{s,s,s})$ also seems complicated, the method in [35] yields some upper estimates.

- If G contains no copy of some fixed tree H on $t + 1$ vertices, then the minimum degree of G is smaller than t. Thus there is a vertex v contained in at most $\binom{t-1}{m}$ copies of K_m, and we can omit it and apply induction to conclude that in this case $ex(n, K_m, H) < t^m n/m!$. It may be that for any such tree the H-free graph G on n vertices maximizing the number of copies of K_m is a disjoint union of cliques all of which besides possibly one are of size t. As mentioned in the beginning of Section 4 this is open even for $H = K_{1,t}$.

- As done for the classical Turán problem of studying the function $ex(n, \mathcal{H})$ for finite or infinite classes $\mathcal{H}$ of graphs, the natural extension $ex(n, T, \mathcal{H})$, which is the maximum number of copies of T in a graph on n vertices containing no member of $\mathcal{H}$, can also be studied. Unlike the case $T = K_2$, there are simple examples here in which $\mathcal{H} = \{H_1, H_2\}$ contains only two graphs, and $ex(n, T, \mathcal{H})$ is much smaller than each of the quantities $ex(n, T, H_1)$ and $ex(n, T, H_2)$. It will be interesting to further explore this behavior.

- Another variant of the problem considered here is that of trying to maximize the number of copies of T in an n-vertex graph, given the number of copies of H in it. The case $H = K_2$ has been studied before, see [1], [28], but the general case seems far more complicated.

- One of the exciting developments in Extremal Combinatorics in recent years has been the study of sparse random analogs of classical combinatorial results, like Turán's Theorem, Ramsey's Theorem, and more. This was initiated in [21] and studied in several papers including [36] and [29], culminating in the papers [14] and [38]. See also [6] and [40] for a more recent effective approach for investigating these problems. The natural sparse random version of the basic problem considered here is the study of the following function. For two graphs H and T with no isolated vertices and for a real $p \in [0, 1]$, let $ex(n, T, H, p)$ be the expected value of the maximum number of copies of the graph T in an H-free subgraph of the random graph $G(n, p)$. Thus $ex(n, T, H, 1)$ is the function $ex(n, T, H)$ studied here. The behavior of $ex(n, T, H, p)$ for $T = K_2$ is quite well understood in many cases, by the results in the papers mentioned above, and it seems interesting to investigate the behavior of the more general function.

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