

Triangles in H -free graphs

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Abstract

For two graphs T and H and for an integer n , let $ex(n, T, H)$ denote the maximum possible number of copies of T in an H -free graph on n vertices. The study of this function when $T = K_2$ is a single edge is the main subject of extremal graph theory. In the present paper we investigate the general function, focusing on the case $T = K_3$, which reveals several interesting phenomena. Three representative results are:

- (i) $ex(n, K_3, H) \leq c(H)n$ iff the 2-core of H is a friendship graph,
- (ii) For any fixed $s \geq 2$ and $t \geq (s-1)! + 1$, $ex(n, K_3, K_{s,t}) = \Theta(n^{3-3/s})$, and
- (iii) $ex(n, K_3, C_5) \leq (1 + o(1)) \frac{\sqrt{3}}{2} n^{3/2}$.

The last statement improves (slightly) an estimate of Bollobás and Győri. The proofs combine combinatorial and probabilistic arguments with simple spectral techniques.

1 Introduction

For two graphs T and H and for an integer n , let $ex(n, T, H)$ denote the maximum possible number of copies of T in an H -free graph on n vertices.

When $T = K_2$ is a single edge, $ex(n, T, H)$ is the well studied function, usually denoted by $ex(n, H)$, specifying the maximum possible number of edges in an H -free graph on n vertices. There is a huge literature investigating this function, starting with the theorems of Mantel [25] and Turán [30] that determine it for $H = K_r$. See, for example, [28] for a survey.

In the present paper we show that the function for other graphs T besides K_2 exhibits several additional interesting features. We illustrate these by focusing on the case of the triangle $T = K_3$, but the question is interesting for other graphs T as well, and many of the results can be extended to other graphs.

There are several sporadic papers dealing with the function $ex(n, T, H)$ for $T \neq K_2$. A notable recent example is given in [20], where the authors determine this function precisely for $T = C_5$ and $H = K_3$. Another example is $T = K_r$ and $H = K_t$ where $r < t$, which follows from the results in [5].

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The case $T = K_3$ and $H = C_{2k+1}$ has also been studied. Bollobás and Györi [7] proved that

$$(1 + o(1)) \frac{1}{3\sqrt{3}} n^{3/2} \leq ex(n, K_3, C_5) \leq (1 + o(1)) \frac{5}{4} n^{3/2}. \quad (1)$$

Györi and Li [18] proved that for any fixed $k \geq 2$

$$\binom{k}{2} ex_{bip}\left(\frac{2n}{k+1}, C_4, C_6, \dots, C_{2k}\right) \leq ex(n, K_3, C_{2k+1}) \leq \frac{(2k-1)(16k-2)}{3} ex(n, C_{2k}), \quad (2)$$

where $ex_{bip}(m, C_4, C_6, \dots, C_{2k})$ denotes the maximum possible number of edges in a bipartite graph on m vertices and girth exceeding $2k$.

As we deal here mainly with $ex(n, K_3, H)$, put $g(n, H) = ex(n, K_3, H)$. Our first result characterizes all graphs H for which $g(n, H) \leq c(H)n$.

The friendship graph F_k is the graph consisting of k triangles with a common vertex. Equivalently, this is the graph obtained by joining a vertex to all $2k$ vertices of a matching of size k . Call a graph an extended friendship graph iff its 2-core is either empty or F_k for some positive k .

Theorem 1.1. *There exists a constant $c(H)$ so that $g(n, H) \leq c(H)n$ if and only if H is a subgraph of an extended friendship graph.*

The next theorem deals with complete bipartite graphs.

Theorem 1.2. *For any fixed $t \geq s \geq 2$ satisfying $t \geq (s-1)! + 1$ there are two constants $c_1 = c_1(s, t)$ and $c_2 = c_2(s, t)$ so that*

$$c_1 n^{3-3/s} \leq g(n, K_{s,t}) \leq c_2 n^{3-3/s}.$$

We also slightly improve the upper estimates in (1) and in (2) above, proving the following.

Proposition 1.3. *The following upper bounds hold.*

- (i) $g(n, C_5) \leq (1 + o(1)) \frac{\sqrt{3}}{2} n^{3/2}$.
- (ii) For any $k \geq 2$, $g(n, C_{2k+1}) \leq \frac{16(k-1)}{3} ex(\lceil n/2 \rceil, C_{2k})$.

A similar result has been proved independently by Füredi and Özkahya [17], who showed that $g(n, C_{2k+1}) \leq 9k ex(n, C_{2k})$.

In addition, we observe that if the chromatic number of H is at least 4 then $g(n, H) \geq c(H)n^3$ for some constant $c(H) > 0$, whereas if the chromatic number of H is at most 3 then $g(n, H) \leq n^{3-\epsilon(H)}$ for some positive constant $\epsilon(H)$.

The proofs are given in the next sections. It is worth noting that the function $g(n, H)$ behaves very differently than its well studied relative $ex(n, H)$. In particular, it is easy to see that for any graph H with at least 2 edges, if $2H$ denotes the vertex disjoint union of two copies of H , then $ex(n, H)$ and $ex(n, 2H)$ have the same order of magnitude. In contrast, if, for example, $H = C_5$ then by (1), $g(n, H) = \Theta(n^{3/2})$ and it is not difficult to show that $g(n, 2H) = \Theta(n^2)$. Similarly, it is known that for any graph H , $ex(n, H)$ is either quadratic in n or is at most $n^{2-\epsilon(H)}$ for some fixed $\epsilon(H) > 0$, whereas it is not difficult to deduce from the results of Ruzsa and Szemerédi in [27] that for the graph H consisting of two triangles sharing an edge $n^{2-o(1)} \leq g(n, H) \leq o(n^2)$.

2 Extended friendship graphs

In this section we prove Theorem 1.1. For a given graph G let $tri(G)$ denote the number of triangles in G . Here and throughout the paper, we often do not make any serious attempt to optimize the absolute constants. We also assume, whenever this is needed, that n is sufficiently large.

We first prove two lemmas.

Lemma 2.1. *Let $G = (V, E)$ be a graph with at least $(9c - 15)(c + 1)n$ triangles and at most n vertices, then it contains a copy of F_c .*

Proof. Take a maximal set of edge-disjoint triangles in G , if they contain a subset of size at least c touching the same vertex then we are done. Otherwise, one can color these triangles with $3(c-2)+1 = 3c - 5$ colors so that no two triangles with the same color share a vertex (by simply coloring each triangle with the smallest available color). Each triangle in our original graph G shares an edge with one of these colored triangles, as they form a maximal set, so there is a set of unicolored triangles with at least $\frac{(9c-15)(c+1)n}{3c-5} = 3(c+1)n$ triangles sharing edges with one of them (where here we are counting the colored triangles too).

Focusing on the triangles colored in this color and the ones sharing edges with them, note that there are at least $3(c+1)n$ of those organized in clusters, with each cluster consisting of one (colored) central triangle and all others sharing an edge with it. There are at most $n/3$ central triangles and hence more than $3cn$ triangles are not central, thus having two vertices in the center and one outside. Call the outside vertex the external one. There are $3cn$ of them, so there must be a vertex $v \in V$ which is an external vertex of at least $3c$ triangles. At most 3 triangles from each cluster can share an external vertex, so there are c triangles from different clusters sharing this vertex, and this is the only vertex they share. These c triangles form a copy of F_c , as needed. \square

Lemma 2.2. *For every $k > 3$ and n large enough there is a graph G on n vertices with at least $\Omega(n^{1+\frac{1}{k-1}})$ triangles and no cycles of length i for any i between 4 and k .*

We note that the exponent here can be slightly improved, at least for some values of k . In particular, for $k = 4$ the best possible value is $(1/6 + o(1))n^{3/2}$, as can be shown using the Erdős-Rényi graph [13], or Theorem 3.3 below with $t = 2$. For our purposes here, however, the above estimate suffices.

Proof. Let G' be a random graph on a fixed set of n labeled vertices obtained by choosing, randomly and independently, each of the $\binom{n}{3}$ potential triangles on the set of vertices to form a triangle in G' with probability $p = \frac{1}{2}n^{-\frac{2k-3}{k-1}}$. Let X be the random variable counting the number of triangles picked, and for $2 \leq i \leq k$ let Y_i denote the random variable counting the number of cycles of length i in which each edge comes from a different triangle. (In particular, Y_2 counts the number of pairs of selected triangles that share two vertices).

Note that if we remove one of our chosen triangles from each such cycle, then the resulting graph will contain no cycle of length between 4 and k . Indeed, if we have such a cycle using two edges of one triangle then replacing those by the third edge will create a shorter cycle, that cannot exist by

assumption. Similarly, a cycle of length 4 cannot be created by two triangles if we leave no pair of triangles sharing two vertices. Put $Z = X - \sum_{i=2}^k Y_i$, and note that it is enough to show that the expectation of Z is at least $\Omega(n^{1+1/(k-1)})$. Indeed, if this is the case, then there is a graph G' for which the value of Z is at least $\Omega(n^{1+1/(k-1)})$. Fixing such a graph and omitting a triangle from each of the short cycles counted by the variables Y_i generates a graph G with the desired properties. Since $\mathbb{E}(X) = \binom{n}{3}p$ and

$$\mathbb{E}(Y_i) = \frac{n \cdot (n-1) \dots (n-i+1)(n-2)^i}{2i} p^i \leq \frac{(n^2 p)^i}{2i} = \frac{n^{i/(k-1)}}{i 2^{i+1}}$$

a simple computation shows that $\mathbb{E}(Z) \geq (1 + o(1))(1/6 - 1/64)n^{1+1/(k-1)}$, as needed. \square

We can now prove Theorem 1.1.

Proof of Theorem 1.1. We start by showing that $g(n, H)$ is linear in n for any extended friendship graph. Let H be an extended friendship graph with h vertices and let G be a graph on n vertices with at least $c(H)n$ triangles, where $c(H) = 10h^2$. We show that G must contain a copy of H .

We first show that G contains a subgraph with minimum degree at least h . As long as there is a vertex in G of degree smaller than h , omit it. This process must terminate with a nonempty graph containing more than $9h^2n$ triangles, since the total number of triangles that can be omitted this way is smaller than $\binom{h}{2}n < h^2n$. We can thus assume that the minimum degree in G is at least h , and that it has at most n vertices and at least $9h^2n$ triangles.

By Lemma 2.1 G contains a copy of the 2-core of H . This copy can be extended to a copy of H . Indeed, if H is disconnected add to it edges to make it connected (keeping the 2-core intact). We can now embed the missing vertices of H in G one by one, starting with the 2-core and always adding a vertex with exactly one neighbor in the previously embedded vertices. Since the minimum degree in G is at least h this can be done, providing the required copy of H .

To complete the proof of the theorem we have to show that if H is not a subgraph of an extended friendship graph then there is a graph G with n vertices and $\omega(n)$ triangles containing no copy of H . Note that H is not a subgraph of an extended friendship graph iff it either contains a cycle of length greater than 3 or it contains two vertex disjoint triangles. In the first case, Lemma 2.2 provides a graph G with a superlinear number of edges containing no copy of H .

For the second case let G be the complete 3-partite graph $K_{1, \lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$. Here all the triangles share a common vertex, hence no two are disjoint. As the number of triangles is $\lfloor \frac{(n-1)^2}{4} \rfloor$, this completes the proof. \square

Remark 2.1. For any connected graph H with h vertices, an n vertex graph consisting of a disjoint union of $\lfloor n/(h-1) \rfloor$ cliques, each of size $h-1$, contains no copy of H and at least $\Omega(h^2n)$ triangles, showing that the estimate in the proof of the last Theorem is tight, up to a constant factor.

3 Complete bipartite graphs

In this section we prove Theorem 1.2 (in a more precise form). We start with an upper bound for the case $H = K_{s,t}$, with $t \geq s$. The case $H = K_{1,t}$ is already taken care of, as this is a tree and hence $g(n, K_{1,t})$ is linear in n . In fact, for this case it is easy to determine $g(n, H)$ much more accurately. If G is a graph on n vertices with no copy of $H = K_{1,t}$ then the maximum degree in G is at most $t - 1$. Therefore, any vertex is contained in at most $\binom{t-1}{2}$ triangles, and hence by double counting the total number of triangles is at most $\frac{1}{3}\binom{t-1}{2}n$. When t divides n this is the precise value of $g(n, K_{1,t})$, as shown by the vertex disjoint union of n/t cliques, each of size t . When t does not divide n a lower bound can be obtained by considering the graph consisting of the disjoint union of $\lfloor n/t \rfloor$ cliques of size t and one clique on the remaining vertices. This is conjectured to give the maximum number of triangles for all values of n and t in [19], where it is shown that this is the case for all $n \leq 2t$.

For $s > 1$ we prove the following.

Lemma 3.1. *For any fixed $1 < s \leq t$*

$$g(n, K_{s,t}) \leq \left(\frac{1}{6} + o(1)\right)(t-1)^{3/s} n^{3-3/s}$$

Proof. If $G = (V, E)$ is $K_{s,t}$ -free then the neighborhood $N(v)$ of any vertex $v \in V$ contains no copy of $K_{s-1,t}$. In [23] Kövari, Sós and Turán prove that for any $t \geq s \geq 2$,

$$ex(m, K_{s,t}) \leq \frac{1}{2}(t-1)^{1/s} m^{2-\frac{1}{s}} + \frac{1}{2}(s-1)m = (1 + o(1))\frac{1}{2}(t-1)^{1/s} m^{2-\frac{1}{s}}.$$

In our setting this gives that the number of edges in the induced subgraph on $N(v)$ satisfies $|E(N(v))| \leq (1 + o(1))\frac{1}{2}(t-1)^{1/(s-1)} d_v^{2-\frac{1}{s-1}}$, where d_v is the degree of v . On the other hand, the number of triangles containing v is exactly $|E(N(v))|$. Therefore:

$$\begin{aligned} tri(G) &\leq (1 + o(1))\frac{1}{6}(t-1)^{1/(s-1)} \sum_v d_v^{2-\frac{1}{s-1}} \\ &\leq (1 + o(1))\frac{1}{6}(t-1)^{1/(s-1)} \left(\sum_v d_v^s\right)^{\frac{2s-3}{s(s-1)}} n^{1-\frac{2s-3}{s(s-1)}} \end{aligned} \quad (3)$$

$$\begin{aligned} &\leq (1 + o(1))\frac{1}{6}(t-1)^{3/s} n^{\frac{2s-3}{(s-1)}} n^{1-\frac{2s-3}{s(s-1)}} \\ &= (1 + o(1))\frac{1}{6}(t-1)^{3/s} n^{3-3/s} \end{aligned} \quad (4)$$

Here we have used Hölder's inequality with $p = \frac{s}{2-\frac{1}{s-1}} = \frac{s(s-1)}{2s-3}$ and $\frac{1}{q} = 1 - \frac{2s-3}{s(s-1)}$ to get the first inequality. To bound the sum $\sum_v d_v^s$ we have used the fact that the number of s -edged stars in G must be at most $\binom{n}{s}(t-1)$ because otherwise t of them share s leaves, creating a $K_{s,t}$. \square

The lower bound is proved by giving appropriate constructions. These shows that the upper bound is tight up to a constant factor for all $t \geq (s-1)! + 1$. For $s = 2$ we can show that even the constant factor is tight.

Lemma 3.2. *For any fixed $s \geq 2$ and $t \geq (s-1)! + 1$ we have $g(n, t) = \Theta(n^{3-\frac{3}{s}})$*

Proof. In view of the previous lemma it suffices to show the existence of a graph G with n vertices containing no copy of $K_{s,t}$ and containing at least $\Omega(n^{3-\frac{3}{s}})$ triangles. For this we apply the projective norm-graphs $H(q, s)$ constructed in [3], where it is proven that these graphs are $K_{s,t}$ free.

The graph $H = H(q, s)$ is defined in the following way: $V(H) = GF(q^{s-1}) \times GF(q)^*$ where $GF(q)^*$ is the multiplicative group of the q element field. For $A \in GF(q^{s-1})$ define the norm

$$N(A) = A \cdot A^q \dots A^{q^{s-2}}.$$

Two vertices (A, a) and (B, b) are connected in H if $N(A + B) = ab$. Note that $|V(H)| = q^s - q^{s-1}$ and H is $q^{s-1} - 1$ regular.

We need to show that $H(q, s)$ has the right number of triangles. The eigenvalues and multiplicities of $H(q, s)$ are given in [29], [4] as follows: $q^{s-1} - 1$ is of multiplicity 1, 0 is of multiplicity $q - 2$, 1 and -1 are of multiplicity $(q^{s-1} - 1)/2$ each, and $q^{(s-1)/2}$, $-q^{(s-1)/2}$ are of multiplicity $(q^{s-1} - 1)(q - 2)/2$ each. Summing the cubes of the eigenvalues we conclude that the number of closed walks of length 3 in $H(q, s)$ is $(q^{s-1} - 1)^3 = (1 + o(1))q^{3s-3}$.

A closed walk of length 3 is not a triangle iff it contains a loop. Fixing $A \in GF(q^t)$ the vertex (A, x) has a loop iff $N(A + A) = x^2$. There are at most 2 solution x for each given A . Thus there are no more than $2q^{s-1}$ loops. A closed walk of length 3 containing a loop must also contain an additional edge taken twice (this additional edge may also be the loop itself). As the graph is $q^{s-1} - 1$ regular we get at most $6q^{s-1}q^{s-1} = o(q^{3s-3})$ such walks containing a loop. As the number of closed walks of length 3 is $(1 + o(1))q^{3s-3}$ this is negligible and the number of triangles is $(\frac{1}{6} + o(1))q^{3s-3} = \Theta(|V(H)|^{3-3/s})$, as needed. \square

Remark 3.1. *For the special case of $s = t = 3$ it can be shown that the construction of Brown [9] gives another example of a $K_{3,3}$ -free on n vertices with essentially the same number of triangles.*

Remark 3.2. *An alternative approach for estimating the number of triangles in the projective norm graphs is to use their pseudo-random properties that follow from the fact that all their nontrivial eigenvalues are much smaller in absolute value than the first. This gives the required estimate for all $s > 3$, where the advantage is that the same argument can be used to estimate the number of copies of bigger cliques or other desired graphs in these graphs. See, for example, [2], Lemma 2.5 for the argument.*

For $s = 2$ we can determine the asymptotic behavior of $g(n, K_{s,t}) = g(n, K_{2,t})$ up to a lower order term, as shown next.

Theorem 3.3. *For any fixed $t \geq 2$, $g(n, K_{2,t}) = (1 + o(1))\frac{1}{6}(t - 1)^{3/2}n^{3/2}$.*

Proof. The upper bound follows from the assertion of Lemma 3.1 with $s = 2$. To prove the lower bound we apply a construction of Füredi [16], extending the one of Erdős and Rényi [13]. The details follow. Let \mathbb{F} be a finite field of order q , where $t - 1$ divides $q - 1$, and let h be a nonzero element of \mathbb{F} that generates a multiplicative subgroup $A = \{h, h^2, \dots, h^{t-1} = 1\}$ of order $t - 1$ in \mathbb{F}^* . The vertices of the graph $G = G(\mathbb{F}, t - 1)$ are all nonzero pairs in $(\mathbb{F} \times \mathbb{F})$, where two pairs (a, b) and (a', b')

are considered equivalent if for some $h^\alpha \in A$, $h^\alpha a = a'$ and $h^\alpha b = b'$. Two vertices $(a, b), (c, d)$ are connected if $ac + bd \in A$. The number of vertices of G is $n = (q^2 - 1)/(t - 1)$ and it is not difficult to check that it is regular of degree q , where here each loop adds one to the degree. Indeed, for a fixed vertex (b, c) and for each $h^\alpha \in A$ there are exactly q solutions (x, y) to the equation $bx + cy = h^\alpha$, and as any neighbor (x, y) of (b, c) is obtained this way $t - 1$ times, by our equivalence relation, the graph is q -regular. Note that there is a (unique) loop at a vertex (x, y) iff $x^2 + y^2 \in A$. For each fixed $h^\alpha \in A$ and each fixed $x \in \mathbb{F}$ there are at most 2 solutions for y , showing that the number of loops is at most $2q(t - 1)/(t - 1) = 2q$ (it is in fact smaller, but this estimate suffices for us).

It thus follows that the number of edges of G (without the loops) is $m = (\frac{1}{2} + o(1))q^3/(t - 1) = (\frac{1}{2} + o(1))\sqrt{t - 1} n^{3/2}$.

We claim that any two distinct vertices (a, b) and (c, d) of G have exactly $t - 1$ common neighbors (if there is a loop in one of these vertices and they are adjacent, this counts as a common neighbor). Indeed, the vertex (x, y) is a common neighbor if for some $0 \leq \alpha, \beta \leq t - 2$

$$\begin{aligned} ax + by &= h^\alpha \\ cx + dy &= h^\beta. \end{aligned}$$

These two equations are linearly independent, and hence there is a unique solution for each choice of α, β . As the number of choices for α and β is $(t - 1)^2$, and every common neighbor is counted this way $t - 1$ times, the claim follows.

By the above claim, G is $K_{2,t}$ -free. In addition, since the endpoints of each edge in it have $t - 1$ common neighbors, each edge is contained in $t - 1$ triangles (including the degenerated ones containing a loop). The number of triangles containing a loop is smaller than $2q^2$ which is far smaller than the number of edges $m = \Theta(q^3/(t - 1))$. Therefore, the number of triangles is

$$(1 + o(1))\frac{1}{3}m(t - 1) = (1 + o(1))\frac{1}{6}(t - 1)\sqrt{t - 1} n^{3/2}$$

completing the proof. □

4 Additional graphs

4.1 Cycles

In this subsection we prove Proposition 1.3, which (slightly) improves the estimates in [7] and [18]. We start with the proof of part (i). Let $G = (V, E)$ be a C_5 -free graph on n vertices with the maximum possible number of triangles. Clearly we may assume that each edge of G lies in at least one triangle. Put $|E| = m$ and $\text{tri}(G) = t$. For each vertex $v \in V$ the graph spanned by its neighborhood $N(v)$ does not contain a path of length 3, and thus, by a known result of Erdős and Gallai [12], the number of edges it spans satisfies $|E(N(v))| \leq d_v$, where $d_v = |N(v)|$ is the degree of v . The number of edges in $N(v)$ is exactly the number of triangles containing v and therefore

$$t \leq \frac{\sum_v d_v}{3} = \frac{2m}{3} \tag{5}$$

Color the vertices of G randomly and independently, where each vertex is blue with probability p (which will be chosen later to be $p = 1/3$) and red with probability $1 - p$. For each edge $e = uv$ of G choose arbitrarily one vertex $w = w(e)$ such that u, v, w form a triangle. Denote by E' the set of edges $e = uv$ of G so that both u and v are colored blue and w is colored red, and denote by V' the set of all blue vertices. Note that the graph (V', E') on the blue vertices contains no C_4 since otherwise each edge of this C_4 forms a triangle together with a red vertex, providing a copy of C_5 in G , which is impossible. Therefore

$$|E'| \leq ex(|V'|, C_4) = \left(\frac{1}{2} + o(1)\right)|V'|^{\frac{3}{2}}.$$

Taking expectation in both sides and using linearity of expectation and the fact that the binomial random variable $|V'|$ is tightly concentrated around its mean we get

$$p^2(1-p)m \leq \mathbb{E}(|E'|) \leq \left(\frac{1}{2} + o(1)\right)(np)^{\frac{3}{2}}.$$

This is because for each edge uv , the probability it belongs to E' is $p^2(1-p)$. Thus

$$m \leq \left(\frac{1}{2} + o(1)\right)n^{\frac{3}{2}} \frac{1}{\sqrt{p(1-p)}}.$$

Since the right hand side is minimized when $p = \frac{1}{3}$ select this p to conclude that

$$m \leq \left(\frac{1}{2} + o(1)\right)n^{\frac{3}{2}} \frac{3\sqrt{3}}{2}.$$

Plugging into (5) we get

$$t \leq \left(\frac{1}{2} + o(1)\right)n^{\frac{3}{2}}\sqrt{3} = \frac{\sqrt{3}}{2}n^{\frac{3}{2}} + o(n^{\frac{3}{2}})$$

as needed. \square

The proof of part (ii) of Proposition 1.3 is similar. Here we do not optimize the value of the probability p and simply take $p = 1/2$, for small values of k the result can be slightly improved. To get the precise statement as stated in the proposition we use an additional trick. The details follow.

Let $G = (V, E)$ be a C_{2k+1} -free graph on n vertices with the maximum possible number of triangles. As before, assume that each edge of G lies in at least one triangle, and for each edge $e = uv$ of G choose a vertex $w = w(e)$ so that u, v, w form a triangle in G . Put $|E| = m$ and $\text{tri}(G) = t$. Since the neighborhood of any vertex v of G contains no path on $2k$ vertices, the Erdős-Gallai theorem implies that it contains at most $(k-1)d_v$ edges, implying that

$$t \leq \frac{\sum_v (k-1)d_v}{3} = \frac{2(k-1)m}{3} \tag{6}$$

Split the vertices of G into $m = \lceil n/2 \rceil$ disjoint subsets, where if n is even each subset is of size 2 and otherwise one subset is of size 1. If a subset chosen is an edge uv of the graph G , we ensure that if $w = w(uv)$ then $u = w(vw)$ and $v = w(uw)$. As the subsets are disjoint, it is easy to check that such a choice is possible. Now color the vertices randomly red and blue: in each subset one vertex is colored red and the other is blue (where each of the two possibilities are equally likely). If n is odd

then the vertex in the last class gets a random color. As before, let E' denote the set of edges $e = uv$ of G so that both u and v are colored blue and $w = w(e)$ is colored red, and denote by V' the set of all blue vertices. The graph (V', E') contains no C_{2k} since otherwise we get a copy of C_{2k+1} in G , which is impossible. Thus

$$|E'| \leq ex(|V'|, C_{2k}) \leq ex(\lceil n/2 \rceil, C_{2k}) \quad (7)$$

since here $|V'|$ is always of cardinality either $\lceil n/2 \rceil$ or $\lfloor n/2 \rfloor$.

We claim that the expected cardinality of E' is at least $m/8$. Indeed, if for an edge uv with $w = w(uv)$ no pair of the three vertices u, v, w lie in a single subset, then the probability that u, v are blue and w is red is exactly $1/8$. For the other edges note that if uv forms one of our subsets and $w = w(uv)$, then the probability that uv lies in E' is 0, but the probability that uw lies in E' is $1/4$ and so is the probability that vw lies in E' . Hence the contribution from these three edges to the expectation of $|E'|$ is $2/4 > 3/8$. Linearity of expectation thus implies that the expected value of $|E'|$ is at least $m/8$ and thus by (7), $m/8 \leq ex(\lceil n/2 \rceil, C_{2k})$, and by (6)

$$t = \text{tri}(G) \leq \frac{16(k-1)}{3} ex(\lceil n/2 \rceil, C_{2k})$$

completing the proof. □

Remark 4.1. *Bondy and Simonovits [8] proved that $ex(n, C_{2k}) \leq O(kn^{1+\frac{1}{k}})$. This has recently been improved by Bukh and Jiang [10] to $ex(n, C_{2k}) \leq O(\sqrt{k \log k} n^{1+\frac{1}{k}})$. Thus the upper bound obtained from the above proof is $g(n, C_{2k+1}) \leq O(k^{3/2} \sqrt{\log k} n^{1+1/k})$.*

4.2 Books

An s -book is the graph consisting of s triangles, all sharing one edge.

Proposition 4.1. *For each fixed $s \geq 2$, if $H = H(s)$ is the s -book then $n^{2-o(1)} \leq g(n, H) = o(n^2)$*

Proof. The lower bound follows from the construction of Ruzsa and Szemerédi [27], based on Behrend's construction [6] of dense subsets of the first n integers that contain no three term arithmetic progressions. This construction gives graphs on n vertices with

$$m = \frac{n^2}{e^{O(\sqrt{\log n})}} = n^{2-o(1)}$$

edges in which every edge is contained in a unique triangle. Therefore these graphs contain no 2-book, and hence no s -book, showing that

$$g(n, H(s)) \geq m/3 \geq \frac{n^2}{e^{O(\sqrt{\log n})}} = n^{2-o(1)}.$$

The upper bound follows from the triangle removal lemma proved in [27]. If G is a graph on n vertices containing t triangles and no copy of $H(s)$, then every edge is contained in at most $s-1$ triangles. Therefore, one has to remove at least $t/(s-1)$ edges of G in order to destroy all triangles. It follows that if $t \geq \epsilon n^2$ then, by the triangle removal lemma, the number of triangles in G is at least δn^3 for some $\delta = \delta(\epsilon, s) > 0$, and thus, by averaging, G contains an r -book for $r \geq 2\delta n > s$, contradiction. Thus $t = o(n^2)$, as needed. □

4.3 Graph Blow-ups

An s blow-up of a graph H is the graph obtained by replacing each vertex v of H by an independent set W_v of size s , and each edge uv of H by a complete bipartite graph between the corresponding two independent sets W_u and W_v .

Proposition 4.2. *Let T be a fixed graph with t vertices. Then $ex(n, T, H) = \Omega(n^t)$ iff H is not a subgraph of a blow-up of T . Otherwise, $ex(n, T, H) \leq n^{t-\epsilon(H)}$ for some $\epsilon(H) > 0$. In particular, $g(n, H) = \Omega(n^3)$ iff the chromatic number of H is at least 4 and otherwise $g(n, H) \leq n^{3-\epsilon(H)}$.*

Proof. If H is not a subgraph of a blow-up of T then the graph G which is the $\ell = \lfloor n/t \rfloor$ -blow-up of T contains no copy of H and yet includes at least $\ell^t = \Omega(n^t)$ copies of T . This establishes the first part of the proposition.

To prove the second part, assume that H is a subgraph of the s -blow-up of T . We have to show that in this case any H -free graph $G = (V, E)$ on n vertices contains less than $n^{t-\epsilon}$ copies of T for some $\epsilon = \epsilon(H) > 0$. Indeed, suppose that G contains m copies of T . Let $V = V_1 \cup V_2 \cup \dots \cup V_t$ be a random partition of V into t pairwise disjoint classes. Let u_1, u_2, \dots, u_t denote the vertices of T . Then the expected number of copies of T in which u_i belongs to V_i for all i is m/t^t . Thus we can fix a partition $V = V_1 \cup V_2 \cup \dots \cup V_t$ so that the number of such copies of T is at least m/t^t . Construct a t -uniform, t -partite hypergraph on the classes of vertices V_1, \dots, V_t by letting a set of vertices v_1, \dots, v_t with $v_i \in V_i$ be an edge iff G contains a copy of T on these vertices, where v_i plays the role of u_i for each i . Therefore, this hypergraph contains at least m/t^t edges. By a well known result of Erdős [11], if the number of edges exceeds $n^{t-\epsilon}$ for an appropriate $\epsilon = \epsilon(t, s) > 0$, then this hypergraph contains a complete t -partite hypergraph with classes of vertices $U_i \subset V_i$, $|U_i| = s$ for all i . This gives an s -blow-up of T in the original graph G , providing a copy of H in it, contradiction. It follows that $m \leq t^t n^{t-\epsilon}$, completing the proof. \square

4.4 Complete graphs

By the argument in [5], or by following one of the well known proofs of Turán's Theorem, it is easy to determine $ex(n, K_r, K_t)$ precisely, for all $r < t$. Indeed, this is exactly the number of copies of K_r in the Turán graph $T(t-1, n)$, which is the complete $t-1$ -partite graph on n vertices with nearly equal color classes. The proof is by observing that if u and v are two non-adjacent vertices in a K_t -free graph, then by making the set of neighbors of u identical to that of v (or vice versa), the graph stays K_t -free, and one can always choose one of these modifications to get a graph containing at least as many copies of K_r as G . Repeating this procedure until every two nonadjacent vertices have the same neighborhoods we get a complete multipartite graph with n vertices and at most $t-1$ color classes, and it is a simple matter to check that the number of copies of K_r in such a graph is maximized when it is the Turán graph $T(t-1, n)$. Thus, in particular, for any $t > 3$,

$$g(n, K_t) = \sum_{0 \leq i < j < k \leq t-2} \lfloor \frac{n+i}{t-1} \rfloor \lfloor \frac{n+j}{t-1} \rfloor \lfloor \frac{n+k}{t-1} \rfloor.$$

5 Extensions

The investigation of the function $ex(n, T, H)$ for general graphs T and H reveals several interesting phenomena. Here we focused mainly on the case $T = K_3$, but the behavior of this function for other graphs T deserves further study. In this section we consider the cases in which T and H are either complete or complete bipartite graphs, extending the results in some of the previous sections. Note that when both T and H are complete graphs the precise value of $ex(n, T, H)$ is known, as mentioned in subsection 4.4. Essentially the same argument suffices to provide the precise value of $ex(n, K_{a,b}, K_t)$. Indeed, if u and v are two non-adjacent vertices in a K_t -free graph G , then by making the set of neighbors of u identical to that of v (or vice versa), the graph stays K_t -free, and one can always choose one of these modifications to get a graph containing at least as many copies of $K_{a,b}$ as G . This is because every copy of $K_{a,b}$ in G that contains both u and v remains a copy in the modified graph as well. Repeating this procedure until every two nonadjacent vertices have the same neighborhoods we get a complete multipartite graph with n vertices and at most $t - 1$ color classes, and one can now optimize the sizes of the color classes to obtain the maximum possible number of copies of $K_{a,b}$. (Note that this optimum is not necessarily obtained for equal or nearly equal color classes).

We next study the function $ex(n, K_m, K_{s,t})$. It is convenient to describe the results in several lemmas.

Lemma 5.1. *For any fixed $m \geq 2$ and $t \geq s \geq m - 1$*

$$ex(n, K_m, K_{s,t}) \leq \left(\frac{1}{m!} + o(1)\right)(t-1)^{\frac{m(m-1)}{2s}} n^{m - \frac{m(m-1)}{2s}}$$

Proof. We apply induction on m . For $m = 2$ the Kövari, Sós Turán result [23] gives $ex(n, K_2, K_{s,t}) = ex(n, K_{s,t}) \leq \left(\frac{1}{2} + o(1)\right)(t-1)^{\frac{1}{s}} n^{2 - \frac{1}{s}}$ and this will serve as our base case. Now assume we have proved this for m and let us prove it for $m + 1$.

In what follows it will be convenient to use the means-inequality: For each $r < s$ and positive reals x_1, \dots, x_n :

$$\sum_{i=1}^n x_i^r \leq n^{1-r/s} \left(\sum_{i=1}^n x_i^s\right)^{r/s}.$$

Let $G = (V, E)$ be a $K_{s,t}$ free graph on n vertices, and let us bound the number of copies of K_{m+1} in it. For each $v \in V$ we know that its neighborhood $N(v)$ does not contain any copy of $K_{s-1,t}$. By the induction assumption we can bound the number of copies of K_m in $N(v)$:

$$\text{num. of } K_m \text{ in } N(v) \leq ex(d_v, K_m, K_{s-1,t}) \leq \left(\frac{1}{m!} + o(1)\right)(t-1)^{\frac{m(m-1)}{2(s-1)}} d_v^{m - \frac{m(m-1)}{2(s-1)}}$$

By bounding the number of K_m in each $N(v)$ we can bound the number of K_{m+1} in G . Denote this number by $\#K_{m+1}$, then

$$\begin{aligned} \#K_{m+1} &\leq \frac{1}{m+1} \left(\frac{1}{m!} + o(1) \right) (t-1)^{\frac{m(m-1)}{2(s-1)}} \sum_v d_v^{m - \frac{m(m-1)}{2(s-1)}} \\ &\leq \left(\frac{1}{(m+1)!} + o(1) \right) (t-1)^{\frac{m(m-1)}{2(s-1)}} \left(\sum_v d_v^s \right)^{\frac{m(2s-m-1)}{2s(s-1)}} n^{1 - \frac{m(2s-m-1)}{2s(s-1)}} \end{aligned} \quad (8)$$

$$\begin{aligned} &\leq \left(\frac{1}{(m+1)!} + o(1) \right) (t-1)^{\frac{(m+1)m}{2s}} n^{\frac{m(2s-m-1)}{2(s-1)} + 1 - \frac{m(2s-m-1)}{2s(s-1)}} \\ &= \left(\frac{1}{(m+1)!} + o(1) \right) (t-1)^{\frac{(m+1)m}{2s}} n^{(m+1) - \frac{(m+1)m}{2s}} \end{aligned} \quad (9)$$

Here we used the means inequality to get the first inequality (an easy calculation shows that $m - \frac{m(m-1)}{2(s-1)} < s$). To bound the sum $\sum_v d_v^s$ we used the fact that the number of s -edged stars in G cannot exceed $\binom{n}{s}(t-1)$ because otherwise t of them will share the same s leaves, creating a $K_{s,t}$. \square

Lemma 5.2. *For any fixed m , $s \geq 2m - 2$ and $t \geq (s-1)! + 1$*

$$ex(n, K_m, K_{s,t}) \geq \left(\frac{1}{m!} + o(1) \right) n^{m - \frac{m(m-1)}{2s}}$$

Proof. We use the projective norm-graphs constructed in [3], where it is shown that $H(q, s)$ is $K_{s, (s-1)!+1}$ free. An (n, d, λ) graph is a d -regular graph on n vertices in which all eigenvalues but the first have absolute value at most λ . A result of the first author (see [22], Theorem 4.10) is the following: Let G_1 be a fixed graph with r edges, s vertices and maximum degree Δ . Let G_2 be an (n, d, λ) graph. If $n \gg \lambda \left(\frac{n}{d} \right)^\Delta$ then the number of copies of G_1 in G_2 is $(1 + o(1)) \frac{n^s}{|Aut(G_1)|} \left(\frac{d}{n} \right)^r$.

In our case we take $G_1 = K_m$ and $G_2 = H(q, s)$. By the results in [29] or [4] we know that the second eigenvalue, in absolute value, of $H(q, s)$ is $q^{\frac{s-1}{2}}$, thus to get the inequality $n \gg \lambda \left(\frac{n}{d} \right)^\Delta$ one must demand that $m < \frac{s+3}{2}$. Plugging our choice of G_1, G_2 into the result mentioned above gives for the number $\#K_m$ of copies of K_m in $H(q, s)$:

$$\begin{aligned} \#K_m &= (1 + o(1)) \frac{n^m}{m!} \left(\frac{1}{q} \right)^{\binom{m}{2}} \\ &= \left(\frac{1}{m!} + o(1) \right) (q^s - q^{s-1})^m \left(\frac{1}{q} \right)^{\binom{m}{2}} \\ &= \left(\frac{1}{m!} + o(1) \right) q^{s(m - \frac{m(m-1)}{2s})} \\ &= \left(\frac{1}{m!} + o(1) \right) n^{m - \frac{m(m-1)}{2s}} \end{aligned}$$

\square

Lemma 5.3. *For any fixed m and $t \geq s \geq 1$ such that $t + s > m$*

$$ex(n, K_m, K_{s,t}) \leq (1 + o(1)) \frac{(m-s)!(t-1)^{\frac{s-1}{2}}}{m!} \binom{t-1}{m-s} n^{\frac{s+1}{2}}$$

Proof. We apply induction on s . As the base case take $s = 1$. In this case the fact that G is $K_{1,t}$ free implies that the degrees of all vertices are at most $t - 1$. Thus each vertex can take part in no more than $\binom{t-1}{m-1}$ copies of K_m and hence

$$ex(n, K_m, K_{1,t}) \leq \frac{1}{m} \binom{t-1}{m-1} n$$

Note that if $t \mid n$ then this bound is achieved by the disjoint union of $\frac{n}{t}$ pairwise vertex disjoint copies of K_t .

Assuming the result for $s - 1$ we prove it for s . If G is $K_{s,t}$ free, then for each $v \in V$ its neighborhood cannot contain a copy of $K_{s-1,t}$. By the induction hypothesis this bounds the number of copies of K_{m-1} by

$$(1 + o(1)) \frac{(m-s)!(t-1)^{\frac{s-2}{2}}}{(m-1)!} \binom{t-1}{m-s} d_v^{\frac{s}{2}}$$

where d_v is the degree of v . This is clearly also the number of copies of K_m containing v . Therefore,

$$\begin{aligned} \#K_m &\leq \frac{1}{m} (1 + o(1)) \sum_v \frac{(m-s)!(t-1)^{\frac{s-2}{2}}}{(m-1)!} \binom{t-1}{m-s} d_v^{\frac{s}{2}} \\ &\leq (1 + o(1)) \frac{(m-s)!(t-1)^{\frac{s-2}{2}}}{m!} \binom{t-1}{m-s} \left(\sum_v d_v^s \right)^{\frac{1}{2}} n^{\frac{1}{2}} \end{aligned} \quad (10)$$

$$\leq (1 + o(1)) \frac{(m-s)!(t-1)^{\frac{s-1}{2}}}{m!} \binom{t-1}{m-s} n^{\frac{s+1}{2}} \quad (11)$$

where we get (10) from the means inequality and (11) from the fact that we cannot have more than $\binom{n}{s}(t-1)$ copies of s stars in G . \square

Note that unlike Lemma 5.1 to get the bound in Lemma 5.3 we need to assume nothing but the obvious fact that K_m does not contain a copy of $K_{s,t}$. On the other hand for every $m, s \in \mathbb{N}$ one has $\frac{s+1}{2} \geq m - \frac{m(m-1)}{2s}$ where we have an equality when $s = m - 1$ and $s = m$. Thus when $s < m - 1$ we must use 5.3, but if $s \geq m - 1$ Lemma 5.1 gives a better upper bound.

Lemma 5.4. *For any m and $t > m - 2 > 1$*

$$ex(n, K_m, K_{2,t}) \geq \frac{1}{4} m^{-\frac{4m}{3}} n^{\frac{4}{3}}$$

Proof. In [24] Lazebnik and Verstraëte show that there exists an m -uniform hypergraph H on n vertices, with at least $\frac{1}{4} m^{-\frac{4m}{3}} n^{\frac{4}{3}}$ hyperedges and with girth at least 5. Let G be the graph obtained from H by replacing each hyperedge of H by a copy of K_m . We next observe that G contains no copy of $K_{2,t}$.

Assume towards a contradiction that G contains a copy of $K_{2,t}$. As $t > m - 2$ the copy of $K_{2,t}$ cannot be contained in a single K_m and so there must be at least two edges in it that come from two different K_m s. These two edges are a part of a C_4 so if we look back at the hypergraph H this C_4 has vertices in at least two hyperedges. Thus H must contain a cycle of length between 2 and 4 in

contradiction to the assumption that H has girth at least 5. From this G is $K_{2,t}$ free with at least $\frac{1}{4}m^{-\frac{4m}{3}}n^{\frac{4}{3}}$ copies of K_m , as needed. \square

Summarizing the results in the last lemmas we conclude that the following holds.

1. For any fixed $m \geq 2$ and $t \geq s \geq m - 1$

$$ex(n, K_m, K_{s,t}) \leq \left(\frac{1}{m!} + o(1)\right)(t-1)^{\frac{m(m-1)}{2s}} n^{m - \frac{m(m-1)}{2s}}$$

2. For any fixed m and $t \geq s \geq 1$ such that $t + s > m$

$$ex(n, K_m, K_{s,t}) \leq (1 + o(1)) \frac{(m-s)!(t-1)^{\frac{s-1}{2}}}{m!} \binom{t-1}{m-s} n^{\frac{s+1}{2}}$$

3. For any fixed m , $s \geq 2m - 2$ and $t \geq (s-1)! + 1$ one has

$$ex(n, K_m, K_{s,t}) \geq \left(\frac{1}{m!} + o(1)\right) n^{m - \frac{m(m-1)}{2s}}$$

Thus, for these values of the parameters

$$ex(n, K_m, K_{s,t}) = \Theta(n^{m - m(m-1)/2s}).$$

4. For all $t > 2$, $ex(n, K_3, K_{2,t}) = (\frac{1}{6} + o(1))(t-1)^{3/2}n^{3/2}$, and for any m and $t > m - 2 > 1$,

$$O(n^{3/2}) \geq ex(n, K_m, K_{2,t}) \geq \frac{1}{4}m^{-\frac{4m}{3}}n^{\frac{4}{3}}$$

We conclude the section by considering the case $T = K_{a,b}$ and $H = K_{s,t}$ where we establish the following.

Proposition 5.5. (i) If $a \leq s \leq t$ and $a \leq b < t$ then

$$ex(n, K_{a,b}, K_{s,t}) \leq (1 + o(1)) \frac{1}{a!(b!)^{1-a/s}} \binom{t-1}{b}^{a/s} n^{a+b-ab/s},$$

and if $a = b$ the above bound can be divided by 2.

(ii) If $(a-1)! + 1 \leq b < (s+1)/2$ then for all $t \geq s$, $ex(n, K_{a,b}, K_{s,t}) = \Theta(n^{a+b-ab/s})$.

Proof. (i) Let $G = (V, E)$ be a $K_{s,t}$ -free graph on n vertices. For each subset B of b vertices, let n_B denote the number of common neighbors of all vertices in B . The number of copies of $K_{a,b}$ in G is clearly exactly $\sum_B \binom{n_B}{a}$ for $b < a$, where the summation here and in what follows is over all b -subsets B of V . If $a = b$ the right hand side should be divided by 2. We proceed with the case $a < b$ recalling that a factor of 1/2 can be added if $a = b$. By the means inequality, the number of copies of $K_{a,b}$ in G is at most

$$\frac{1}{a!} \sum_B n_B^a \leq \frac{1}{a!} \binom{n}{b}^{1-a/s} \left(\sum_B n_B^s\right)^{a/s}$$

$$\leq (1 + o(1)) \frac{n^{b-ab/s}}{a!(b!)^{1-a/s}} \binom{t-1}{b} n^s)^{a/s} = (1 + o(1)) \frac{1}{a!(b!)^{1-a/s}} \binom{t-1}{b}^{a/s} n^{a+b-ab/s}.$$

Here we used the fact that $\sum_B n_B^s \leq (1 + o(1)) \binom{t-1}{b} n^s$ since if we have more than $\binom{t-1}{b}$ subsets of cardinality b in V with each of them having the same s -subset among their common neighbors, then we get a copy of $K_{s,t}$, which is impossible.

(ii) The projective norm graphs give, as in the proof of Lemma 5.2, that if $(a-1)! + 1 \leq b < (s+1)/2$ then $ex(n, K_{a,b}, K_{s,t}) \geq \Omega(n^{a+b-ab/s})$. This and part (i) supply the assertion of part (ii). \square

6 Concluding remarks and open problems

- We have studied the function $ex(n, T, H)$ focusing on the investigation of $g(n, H) = ex(n, K_3, H)$. Even in this special case there are many difficult problems that remain open. One such problem that received a considerable amount of attention is the case that H is the 2-book, that is, two triangles sharing an edge. This is equivalent to the problem of obtaining tight bounds for the triangle removal lemma, which is wide open despite the fact we know that here $n^{2-o(1)} \leq g(n, H) \leq o(n^2)$ and despite some recent progress in [15]. The classical function $ex(n, H)$ is determined, up to a low order term, by the Erdős-Stone Theorem [14] for any H with chromatic number at least 3. The determination of $g(n, H)$ is more complicated, and we do not even know its correct order of magnitude for some simple 3-chromatic graphs like odd cycles. In this specific case, however, it may be that the lower bound in (2) and the upper bound in Proposition 1.3 differ only by a constant factor, as it may be true that the functions $ex(m, C_{2k})$ and $ex_{bip}(m, C_4, C_6, \dots, C_{2k})$ differ only by a constant factor. The problem of determining the correct order of magnitude of $g(n, K_{s,s,s})$ also seems complicated, the method in [26] yields some upper estimates.
- If G contains no copy of some fixed tree H on $t+1$ vertices, then the minimum degree of G is smaller than t . Thus there is a vertex v contained in at most $\binom{t-1}{2}$ triangles, and we can omit it and apply induction to conclude that in this case $g(n, H) < t^2 n/2$. It may be that for any such tree the H -free graph G on n vertices maximizing the number of triangles is a disjoint union of cliques all of which besides possibly one are of size t . As mentioned in the beginning of Section 3 this is open even for $H = K_{1,t}$.
- As done for the classical Turán problem of studying the function $ex(n, \mathcal{H})$ for finite or infinite classes \mathcal{H} of graphs, the natural extension $ex(n, T, \mathcal{H})$, which is the maximum number of copies of T in a graph on n vertices containing no member of \mathcal{H} , can also be studied. Unlike the case of graphs, there are simple examples here in which $\mathcal{H} = \{H_1, H_2\}$ contains only two graphs, and $ex(n, T, \mathcal{H})$ is much smaller than each of the quantities $ex(n, T, H_1)$ and $ex(n, T, H_2)$. It will be interesting to further explore this behavior.
- Another variant of the problem considered here is that of trying to maximize the number of copies of T in an n -vertex graph, given the number of copies of H in it. The case $H = K_2$ has

been studied before, see [1], [21], but the general case seems far more complicated.

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