Induced universal hypergraphs

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Abstract

We prove that the minimum number of vertices of a hypergraph that contains every d-uniform hypergraph on k vertices as an induced sub-hypergraph is $(1+o(1))\,2^{\binom{k}{d}/k}$. The proof relies on the probabilistic method and provides a non-constructive solution. In addition we exhibit an explicit construction of a hypergraph on $\Theta\left(2^{\binom{k}{d}/k}\right)$ vertices, containing every d-uniform hypergraph on k vertices as an induced sub-hypergraph.

1 Introduction

For a fixed $d \geq 3$ and given $k > 0$, we study the smallest possible number of vertices of a d-uniform hypergraph H that contains every d-uniform hypergraph on k vertices as an induced sub-hypergraph. Call such a hypergraph a *d-uniform k-induced-universal hypergraph.* If $n = |V(H)|$, then

$$n \geq (1+o(1))\,2^{\binom{k}{d}/k}$$

as there are $\frac{n!}{(n-k)!}$ induced labelled sub-hypergraphs of H of size k, and there are $2^{\binom{k}{d}}$ different labelled d-uniform hypergraphs of size k, so for H to be k-induced-universal, the following inequality must hold:

$$n = (1+o(1))\left(\frac{n!}{(n-k)!}\right)^{1/k} \geq (1+o(1))\,2^{\binom{k}{d}/k}$$

(The equality holds for our parameters, since n is much larger than k).

In Theorem 3.2, we construct for every $d \geq 3$ an explicit d-uniform k-induced-universal hypergraph whose number of vertices is at most

$$(2.89+o(1))\,2^{\binom{k}{d}/k}.$$

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A result for $d = 2$ was provided by [4], in which the number of vertices in the induced universal graph is no more than

$$(22.63 + o(1))\, 2^{\binom{k}{2}/k}.$$

In Theorem 4.6 we prove the existence of a d-uniform k-induced-universal hypergraph of size $(1 + o(1))\, 2^{\binom{k}{d}/k}$ for $d \geq 3$. This is optimal up to the $o(1)$ term, and extends the proof for $d = 2$ which appears in [2].

Universal graphs and hypergraphs can be defined for specific families as well. For a family $\mathcal{H}$ of hypergraphs, a hypergraph H is $\mathcal{H}$-induced-universal if it contains every member of $\mathcal{H}$ as an induced sub-hypergraph.

There is a vast literature regarding the minimum possible size of induced universal graphs for finite families of graphs [2 - 13, 15]. However, not much is known regarding induced universal *hypergraphs*. In the present paper, we discuss this problem.

The paper is organized as follows. In Section 2 we define the concept of a *system of predicates* over a family of hypergraphs, which will be used to force a labelling on the vertices of the hypergraphs. We describe a general method for constructing explicit induced universal hypergraphs for such families. The stricter the predicates are, the more information we gain by forcing the labelling, and the smaller the constructed induced universal hypergraphs are. In Section 3 we apply this method using a specific structure, to generate a d-uniform k-induced-universal hypergraph. Section 4 contains a proof of the existence of an asymptotically optimal d-uniform k-induced-universal hypergraph, by combining probabilistic arguments with the results of Section 2. The final Section 5 contains some concluding remarks and open problems.

Throughout the paper, we assume that $d \geq 3$ is fixed and $k > 0$ tends to infinity. We omit all floor and ceiling signs whenever these are not crucial. All logarithms are in base 2 unless otherwise specified.

2 Hypergraph predicates

We start with an outline of the method that we call in what follows the method of *hypergraph predicates.* Note, first, that if there is a way to number the vertices of each hypergraph G in a given family $\mathcal{H}$ of d uniform hypergraphs on k vertices by $v_1, v_2, \ldots, v_k$ and then assign to each vertex v_i one of L_i possible labels, so that for every d-tuple of vertices it is possible to decide whether or not they form an edge of G from their indices and the values of the labels, then there is an induced universal hypergraph for $\mathcal{H}$ whose number of vertices is $\sum_{i=1}^{k} L_i$. Indeed this is simply the hypergraph analogue of the method of [10]. Given labels as above the induced universal hypergraph consists of pairwise disjoint sets V_i where V_i is of size L_i and its members correspond to the labels in L_i. A d-tuple of vertices in distinct sets V_i forms an edge iff the corresponding vertices with these labels form an edge in the decision procedure above.

It thus suffices to describe an economical labeling. Split the set of all $\binom{k}{d}$ potential edges into pairwise disjoint sets, where all edges in each such set have a common vertex which is assigned to them. For each such set s of potential edges we will have a predicate P describing which of the $2^{|s|}$ possible assignments for these potential edges, (that is, which of them are edges of G and which are nonedges), are allowed. When the number of such possibilities, say M_s, is significantly smaller than $2^{|s|}$, then the fact that the predicate is satisfied will supply some saving in the size of the labels. The specific possibility among the M_s possible ones will be part of the label of the vertex assigned to the set s. The idea is to define the disjoint sets of potential edges, the vertices assigned to them and the predicates in such a way that all labels L_i will be nearly equal, ensuring that each member G of our family of hypergraphs has a vertex labeling so that all predicates are satisfied. This will supply the required induced universal hypergraph.

As a simple example indicating how the above method can yield nontrivial saving consider the case of graphs ($d = 2$). In this case among any three potential edges incident with a single vertex, say vertex number 1, either two are edges of G or two are nonedges of G. By numbering the vertices appropriately we can thus always ensure that for the pair of edges 12 and 13 there are only two possibilities (and not $2^2 = 4$) - either both are edges or both are nonedges. The corresponding predicate in this case is the one that allows only the two options above, and one bit in the label of the first vertex suffices to encode which of the two options indeed holds.

The general case follows this outline but uses more sophisticated predicates. An argument based on Hall's Theorem shows we can ensure that all labels are nearly equal. We proceed with the formal details.

We start by defining several properties of a system of predicates over variables denoting hyperedges. We provide a generic theorem (Theorem 2.7) for using such a system in order to build an explicit small induced universal hypergraph. We also describe a useful lemma (Lemma 2.8), which we use commonly throughout this paper, providing a simple condition for applying the generic theorem.

Definition 2.1. Denote by $\widehat{G}_{d,k}$ the *generic hypergraph*, a d-uniform hypergraph on k vertices whose edges are defined as variables in $\mathbb{Z}_2$. That is, if a hypergraph G' is an evaluation of $\widehat{G}_{d,k}$, $e_T = 1$ iff the hyperedge on the set T is in $E(G')$.

Definition 2.2. Let $\widehat{G} = \widehat{G}_{d,k}$ and let S be a system of predicates on $E(\widehat{G})$. Each predicate $\langle P, s\rangle \in S$ is of the form $P : 2^s \to \{0, 1\}$, where s is a set of edges, and P is a predicate on the edges. Thus, P gives a truth value of 0 or 1 to each possible assignment for the edges in s. A set $s \subset E(\widehat{G})$ is called *forced* by S if $\exists P : \langle P, s\rangle \in S$.

Definition 2.3. A system of predicates S on $E(\widehat{G})$ is called *assignable* if:

1. For every two predicates $\langle P_1, s_1\rangle, \langle P_2, s_2\rangle \in S$, $s_1 \cap s_2 = \emptyset$. That is, the predicates act on pairwise disjoint sets of edges.

2. For every predicate $\langle P, s\rangle \in S$, $\bigcap_{e\in s} e \neq \emptyset$. That is, there is at least one vertex in $V(\widehat{G})$, common to all edges in s.

Denote by $F_S = \left\{e \in E(\widehat{G}) : \nexists \langle P, s\rangle \in S, e \in s\right\}$ the set of *free* edges, namely, edges which are not in any set s forced by S. Denote by $C_S = \{s : \langle P, s\rangle \in S\} \cup \{\{e\} : e \in F_S\}$ the set of dependent components of S. Notice that C_S is a partition of $E(\widehat{G})$. For a set of edges $s \in C_S$, denote by $R(s) = \bigcap_{e\in s} e \neq \emptyset$ the *representatives* of s. A function $f : C_S \to V(\widehat{G})$ is called an *assignment* of S if for every $s \in C_S$, $f(s) \in R(s)$. Similarly, a *partial assignment* is a an assignment of values of $f(s)$ for some $s \in C_S$.

Definition 2.4. For $s \in C_S$, let the *solution space* of s be the set of possible assignments for the edges in s. If s is forced by S with a predicate P, consider only the assignments giving a truth value for P. If s is not forced by S, then s is a single edge, and the solution space is simply the two possible assignments for the edge. Denote the *entropy* of s by $\text{ent}(s) = \log|\text{Solution Space } of\ s|$, and the *information* of s by $\text{info}(s) = |s| - \text{ent}(s)$. Finally, denote $\sum_{s\in C_S} \text{info}(s)$ by $\text{info}(S)$.

Remark. When the component $s = \{e\}$ is not forced by any predicate, then $\text{ent}(s) = 1$ and $\text{info}(s) = 0$. In a common case in this paper, a predicate $\langle P, s\rangle$ on a set $s = \{e_1, e_2\}$ will state that $e_1 = e_2$. In this case, $\text{ent}(s) = 1$ and $\text{info}(s) = 1$.

In order to construct a small k-induced-universal hypergraph, we would like to find efficient assignments. The following definition helps quantifying the efficiency of assignments.

Definition 2.5. Let f be an assignment (or a partial assignment) of S, and let $v \in V(\widehat{G})$. Let

$$W_f(v) = \sum_{s\in f^{-1}(v)} \text{ent}(s) = \sum_{s\in C_S : f(s)=v} \text{ent}(s)$$

be the *weight* of v under f. Note that when f is an assignment (not a partial one),

$$\sum_{v\in V(\widehat{G})} W_f(v) = \sum_{s\in C_S} \text{ent}(s) = \sum_{s\in C_S} |s| - \text{info}(s) = \binom{k}{d} - \text{info}(S).$$

An assignment f is called a λ-*fair assignment* of S if

$$\max_{v\in V(\widehat{G})} W_f(v) - \min_{v\in V(\widehat{G})} W_f(v) \leq \lambda.$$

Call f a *perfect assignment* if it is a 0-fair assignment. Note that in this case:

$$W_f(v) = \frac{\binom{k}{d} - \text{info}(S)}{k}$$

for every $v \in V(\widehat{G})$. When $\lambda = 1$, we simply call f a *fair assigment.*

Remark. Finding a λ-fair assignment in general is $NP-hard$, therefore there is no simple necessary and sufficient condition for the existence of such an assignment. However, we deal mostly with the case where $\mathrm{ent}\,(s)=1$ for every s, except maybe a constant number of components, and look for a 1-fair assignment. This problem is polynomially solvable and there is a simpler criterion for verifying that S is fairly assignable, as we show later in Lemma 2.8.

We now use the properties defined for a system of predicates, and apply them to get an induced universal hypergraph covering some family of hypergraphs.

Definition 2.6. Let $\mathcal{H}$ be a family of d-uniform hypergraphs on k vertices, and let S be a system of predicates. Say that S *describes* $\mathcal{H}$ if for every hypergraph $H\in\mathcal{H}$, there exists a labelling of $V(H)$ such that all the predicates in S hold on the labelled edges of H. Denote the family of all d-uniform hypergraphs on k vertices by $\mathcal{H}_{d,k}$.

The main theorem of this section is the following:

Theorem 2.7. *Let $\mathcal{H}$ be a family of d-uniform hypergraphs on k vertices, let S be an assignable system of predicates describing $\mathcal{H}$, and let f be an assignment of S.*

1. *There exists an induced universal hypergraph for $\mathcal{H}$ of size $\sum_{v\in V(\widehat{G})} 2^{W_f(v)}$.*
2. *If f is a perfect assignment, then there exists an induced universal hypergraph for $\mathcal{H}$ of size $k\cdot 2^{\frac{\binom{k}{d}-\mathrm{info}(S)}{k}}$.*
3. *If f is a λ-fair assignment for $\lambda>0$, then there exists an induced universal hypergraph for $\mathcal{H}$ of size $C_\lambda k\cdot 2^{\frac{\binom{k}{d}-\mathrm{info}(S)}{k}}$ where*

$$C_\lambda=\frac{\left(2^\lambda-1\right)2^{\lambda/\left(2^\lambda-1\right)}\log e}{e\lambda}$$

   is a constant, depending only on λ.

Proof. For a vertex $v\in V(\widehat{G})$, let $C_f\,(v)=\bigcup f^{-1}\,(v)$ be the set of edges assigned to v, and let $T_f\,(v)$ be the solution space of $C_f\,(v)$, that is, the space of all possible evaluations of the edges, satisfying all the relevant predicates. Note that

$$\log\left|T_f\,(v)\right|=\sum_{s\in f^{-1}(v)}\mathrm{ent}\,(s)=W_f\,(v)\,.$$

Define the induced universal hypergraph H as follows:

$$V(H)=\left\langle v,\vec{t}\right\rangle$$

where $v\in V(\widehat{G})$ and $\vec{t}\in T_f\,(v)$. Let $e=\left\{\left\langle v_1,\vec{t_1}\right\rangle,\ldots,\left\langle v_d,\vec{t_d}\right\rangle\right\}$ be a set of d vertices of H, and let $e'=\{v_1,\ldots,v_d\}$. If $|e'|<d$, then we say that $e\notin E\,(H)$. Otherwise, $e'\in E(\widehat{G})$, let s be the component in C_S containing e'. Put $v=f\,(s)$.

Since f is an assignment, and $e' \in s$, $v = v_i \in e'$ for a unique i, $1 \le i \le d$. Since $\vec{t_i} \in T_f(v_i)$, the value of the variable e' is determined in the solution vector. We say that $e \in E(H)$ iff the value of e' in $\vec{t_i}$ is 1.

Note that indeed $|V(H)| = \sum_{v \in V(\widehat{G})} 2^{W_f(v)}$, simply because $|T_f(v)| = 2^{W_f(v)}$ for every $v \in V(\widehat{G})$.

Next, we show that H is indeed induced universal for $\mathcal{H}$. Let $G \in \mathcal{H}$. Since S describes $\mathcal{H}$, let $V(G) = \{v_1, \ldots, v_k\}$ be a labelling of the vertices of G satisfying all the predicates of S. For each v_i, let $\{e_{i,1}, \ldots, e_{i,r_i}\}$ be the set of edges in $\widehat{G}$ for which their component is mapped by f to v_i. Since the predicates of S hold for these variables, $(e_{i,1}, \ldots, e_{i,r_i})$ equals a vector $\vec{t_i} \in T_f(v_i)$. The sub-hypergraph $\{\langle v_1, \vec{t_1} \rangle, \ldots, \langle v_k, \vec{t_k} \rangle\}$ of H is therefore isomorphic to G.

The results for a perfect assignment are immediate. Suppose f is a λ-fair assignment. Put $n = \sum_{v \in V(\widehat{G})} W_f(v) = \binom{k}{d} - \operatorname{info}(S)$ and consider the vector $\vec{w} = (w_1, \ldots, w_k)$, where $w_i = W_f(v_i)$. The vector $\vec{w}$ resides in the convex set defined by $w_i \ge 0$, $\sum_{i=1}^{k} w_i = n$, and $\max w_i - \min w_i \le \lambda$, and we get that $\phi(\vec{w}) = |V(H)| = \sum_{i=1}^{k} 2^{w_i}$. The function ϕ is maximized when $\vec{w}$ is on the boundary of the set where $\phi(\vec{w}) \le \alpha k \cdot 2^{n/k - \alpha\lambda + \lambda} + (1-\alpha)\, k \cdot 2^{n/k - \alpha\lambda}$ for some $0 \le \alpha \le 1$. Therefore

$$|V(H)| \le \left[\alpha \cdot 2^{-\alpha\lambda+\lambda} + (1-\alpha) \cdot 2^{-\alpha\lambda}\right] k 2^{n/k}.$$

The expression $\alpha \cdot 2^{-\alpha\lambda+\lambda} + (1-\alpha)\, 2^{-\alpha\lambda}$ is maximized when

$$\alpha = \frac{2^\lambda - 1 - \lambda \ln 2}{(2^\lambda - 1)(\lambda \ln 2)},$$

and its maximal value is

$$\frac{\left(2^\lambda - 1\right) 2^{\lambda/\left(2^\lambda - 1\right)} \log e}{e\lambda}.$$

□

Remark. For $\lambda = 1$, $C_1 = \frac{2 \log e}{e} \approx 1.06$.

In order to find a small induced universal hypergraph, we try to find a λ-fairly assignable system of predicates S describing a given family of hypergraphs, maximizing $\operatorname{info}(S)$, and minimizing λ. The following lemma will be used extensively throughout the rest of the paper.

Lemma 2.8. *Let S be an assignable system of predicates and suppose $|F_S| > \binom{k}{d} \cdot \left(1 - \frac{1}{2k}\right)$ and $k \ge 2^d \cdot d!$. Then S is fairly assignable.*

Proof. Start by letting f' be a partial assignment of S, assigning all non trivial components to one of their representatives arbitrarily. Our goal is to extend f' to a fair assignment f. Denote the weight of v under f' by $W_{f'}(v)$. Let $c : V(\widehat{G}) \to \mathbb{Z}$ be a function such that $\sum_{v \in V(\widehat{G})} c(v) = |F_S|$, minimizing $\sum_{v \in V(\widehat{G})} (W_{f'}(v) + c(v))^2$. Call c the *capacity* function of f.

Claim 2.9. Put $t_c(v) = W_{f'}(v) + c(v)$. The following inequality holds:

$$\max_{v \in V(\widehat{G})} t_c(v) - \min_{v \in V(\widehat{G})} t_c(v) \leq 1.$$

Proof. Otherwise, there exist vertices $u, w \in V(\widehat{G})$ such that $t_c(u) - t_c(w) > 1$. Let c' be a modification of c achieved by incrementing $c(w)$ by 1 and decrementing $c(u)$ by 1. Clearly $c' : V(\widehat{G}) \to \mathbb{Z}$ and $\sum_{v \in V(\widehat{G})} c'(v) = |F_S|$.

$$\begin{aligned}
&\sum_{v \in V(\widehat{G})} t_{c'}(v)^2 = \sum_{v \in V(\widehat{G}) \setminus \{u,w\}} t_c(v)^2 + t_{c'}(u)^2 + t_{c'}(w)^2 = \\
&\sum_{v \in V(\widehat{G})} t_c(v)^2 - t_c(u)^2 + (t_c(u) - 1)^2 - t_c(w)^2 + (t_c(w) + 1)^2 = \\
&\sum_{v \in V(\widehat{G})} t_c(v)^2 - 2t_c(u) + 2t_c(w) + 2 < \sum_{v \in V(\widehat{G})} t_c(v)^2,
\end{aligned}$$

contradicting minimality. □

Claim 2.10. For every $v \in V(\widehat{G})$, $c(v) \geq 0$.

Proof. Notice that:

$$\begin{aligned}
&\sum_{v \in V(\widehat{G})} W_{f'}(v) + c(v) = \sum_{\langle P,s \rangle \in S} \mathrm{ent}(s) + |F_S| = \\
&\sum_{\langle P,s \rangle \in S} (|s| - \mathrm{info}(s)) + |F_S| = \left| \bigcup_{\langle P,s \rangle \in S} s \right| - \mathrm{info}(S) + |F_S|.
\end{aligned}$$

As the edges in $\left(\bigcup_{\langle P,s \rangle} s\right) \cup F_S$ are simply all the edges in $E(\widehat{G})$, we get that $\sum_{v \in V(\widehat{G})} W_{f'}(v) + c(v) = \binom{k}{d} - \mathrm{info}(S)$. Together with the fact that

$$\max_{v \in V(\widehat{G})} W_{f'}(v) + c(v) - \min_{v \in V(\widehat{G})} W_{f'}(v) + c(v) \leq 1,$$

we conclude that for every $v \in V(\widehat{G})$, $W_{f'}(v) + c(v) \geq \frac{\binom{k}{d} - \mathrm{info}(S)}{k} - 1$.

Since $|F_S| > \binom{k}{d} \cdot \left(1 - \frac{1}{2k}\right)$, the number of edges in $E(\widehat{G}) \setminus F_S$ is less than $\binom{k}{d}/2k$, and therefore $\mathrm{info}(S) < \binom{k}{d}/2k < \binom{k}{d}/2$ and also $\sum_{v \in V(\widehat{G})} W_{f'}(v) < \binom{k}{d}/2k$.

In particular $W_{f'}(v) + c(v) \geq \frac{\binom{k}{d} - \mathrm{info}(S)}{k} - 1 \geq \binom{k}{d}/2k - 1$ and $W_{f'}(v) < \binom{k}{d}/2k$. Therefore $c(v) > -1$ and since $c(v) \in \mathbb{Z}$, $c(v) \geq 0$. □

We next show that it is possible to extend f' to f so that for each $v \in V$, $|\{e \in F_S : f(\{e\}) = v\}| = c(v)$. To do so, we use a weighted version of Hall theorem:

Theorem (Hall)**.** Let B be a bipartite graph on two sides X and Y, and let $c : Y \to \mathbb{N}$ be a capacity function on Y, such that $\sum_{y\in Y} c(y) = |X|$. If for every $A \subset X$, $\sum_{y\in N(A)} c(y) \geq |A|$, then there exists a subgraph $B' \subset B$, $V(B') = V(B)$, in which for every $x \in X$, $d_{B'}(x) = 1$ and for every $y \in Y$, $d_{B'}(y) = c(y)$.

Define a bipartite graph B on two sides $X = F_S$ and $Y = V(\widehat{G})$, using the capacity function c, where each edge $x \in X$ is connected to all $v \in Y$ participating in this edge. Indeed, given a subgraph $B' \subset B$ as above, it is possible to extend f' to an assignment f such that $f(\{e\}) = v$ if $e \in X$ and $v \in Y$ are connected in B'. By its definition, $f : C_S \to V(\widehat{G})$ and $W_f(v) = W_{f'}(v) + c(v)$ and as a result, f is a fair assignment.

Let $A \subset X$, and let $r = |N(A)|$. If $r = k$ then $\sum_{y\in N(A)} c(y) = |F_S| \geq |A|$, and if $r = 0$ then $A = \emptyset$. Assume $1 \leq r \leq k-1$. On the one hand, $|A| \leq \binom{r}{d}$ as this is the maximum number of edges which can be generated using only r vertices. On the other hand,

$$\begin{aligned}
\sum_{y\in N(A)} c(y) &= \sum_{y\in N(A)} \left(W_{f'}(y) + c(y) - W_{f'}(y)\right) \\
&\geq r\left(\frac{\binom{k}{d} - \mathrm{info}(S)}{k} - 1\right) - \sum_{y\in Y} W_{f'}(y) \\
&\geq \frac{r}{k}\left(\binom{k}{d} - \mathrm{info}(S)\right) - r - \left|E(\widehat{G})\right| + |F_S| + \mathrm{info}(S) \\
&\geq \frac{r}{k}\cdot\binom{k}{d} - r - \left|E(\widehat{G})\right| + |F_S| \\
&\geq \frac{r}{k}\cdot\binom{k}{d} - r - \frac{\binom{k}{d}}{2k}.
\end{aligned}$$

In order to use Hall's theorem, it suffices to show that

$$\frac{r}{k}\cdot\binom{k}{d} - r - \frac{\binom{k}{d}}{2k} \geq \binom{r}{d}$$

or equivalently, that

$$r\left(\binom{k}{d}/k - \binom{r}{d}/r\right) \geq r + \frac{\binom{k}{d}}{2k}.$$

Notice that since $1 \leq r \leq k-1$,

$$\begin{aligned}
\binom{k}{d}/k - \binom{r}{d}/r &= \frac{(k-1)\ldots(k-d+1) - (r-1)\ldots(r-d+1)}{d!} \\
&\geq \frac{(k-r)(k-2)\ldots(k-d+1)}{d!} = \frac{k-r}{k-1}\cdot\frac{\binom{k}{d}}{k},
\end{aligned}$$

and therefore

$$r\left(\binom{k}{d}/k-\binom{r}{d}/r\right)\geq\frac{r\,(k-r)}{k-1}\cdot\frac{\binom{k}{d}}{k}\geq\frac{\binom{k}{d}}{k}.$$

On the other hand, when $k\geq 2^d\cdot d!$ and $d\geq 3$, then

$$r\leq k\leq\frac{k^2}{2^d\cdot d!}\leq\left(\frac{k}{2}\right)^{d-1}\cdot\frac{1}{2d!}\leq\frac{\binom{k}{d}}{2k}.$$

And therefore

$$r+\frac{\binom{k}{d}}{2k}\leq\frac{\binom{k}{d}}{k}.$$

□

3 Information-wise optimal predicate

In this section, we show a predicate describing all hypergraphs for $d\geq 3$, which maximizes the amount of information achieved. We show that the maximal amount of information in any such predicate is bounded by $\log k!$, and indeed show here a predicate providing $\log k!-o\,(k)$ bits of information. By showing it is also fairly assignable, we generate an explicit d-uniform k-induced-universal hypergraph on roughly $2.89\cdot 2^{\binom{k}{d}/k}$ vertices.

The following simple claim bounds the maximum amount of information in a (not necessarily assignable) system of predicates describing $\mathcal{H}=\mathcal{H}_{d,k}$ by roughly $k\log k-\log_2 e\cdot k$.

Claim 3.1. Let S be a system of predicates describing $\mathcal{H}$. Then $\text{info}\,(S)\leq\log k!=k\log k-c'k+o\,(k)$ where $c'=\log_2 e\approx 1.44$.

Proof. Let S be a system of predicates, and suppose $\text{info}\,(S)>\log k!$. Let $G=G\,(k,d,0.5)$ be the random d-uniform hypergraph on k vertices obtained by picking each d-tuple randomly and independently to be an edge with probability $1/2$, and let $\sigma\in S_k$ be some permutation. For a component $s\in C_S$, the probability that under the labelling σ of G, the evaluation of s is indeed in the solution space of s is

$$\frac{|\text{Solution Space}|}{2^{|s|}}=2^{\text{ent}(s)-|s|}=2^{-\text{info}(s)}.$$

Since the evaluations of the edges in each component are independent of each other, the probability that all the predicates in S are satisfied for the labelling σ of G is

$$\prod_{s\in C_S}2^{-\text{info}(s)}=2^{-\text{info}(S)}<1/k!.$$

Therefore, there exists a hypergraph G for which there exists no permutation σ satisfying all the predicates. □

Next, we show that indeed there exists a fairly assignable system of predicates describing $\mathcal{H}_{d,k}$ providing $\log(k!) - o(k)$ bits of information.

Theorem 3.2. *Let $d \geq 3$. There exists a system of one predicate $S = \{\langle P, s\rangle\}$, describing $\mathcal{H}_{d,k}$, where S is a fairly assignable predicate, and* $\text{info}(S) = k\log k - k\log e + o(k)$.

Proof. Let $V(\widehat{G}) = U \cup L \cup R$ where U is a set of $m = \lceil 2\log k\rceil$ vertices, L is a disjoint set of $l = d-2 \geq 1$ vertices and R is the remaining set of $r = k-m-l$ vertices. Denote the vertices of R by $\{v_1,\ldots,v_r\}$ and the vertices of U by $\{u_0,\ldots,u_{m-1}\}$. For a vertex $v \in R$, let $\phi(v) = \sum_{i=0}^{m-1} e_{L\cup\{v,u_i\}} \cdot 2^i$ (notice that $|L\cup\{v,u_i\}| = d$ and recall that $e_{L\cup\{v,u_i\}} \in \{0,1\}$). Let $s = \left\{e_{L\cup\{v,u_i\}} : v \in R, 0 \leq i < m\right\}$, and define $P : 2^s \to \{0,1\}$ to be the predicate stating that $\phi(v_i) \leq \phi(v_j)$ whenever $i \leq j$. Thus, P holds iff the vertices of R are sorted according to their links on U and L.

We conclude the proof using the next four simple claims.

Claim 3.3. S is assignable.

Proof. The vertices of L are common to all the edges used by s. □

Claim 3.4. S describes $\mathcal{H}_{d,k}$.

Proof. Set aside the vertices L and U, and calculate $\phi(v)$ for each remaining vertex $v \in R$. Label the vertices of R so that ϕ is monotonically non-decreasing. □

Claim 3.5. $\text{info}(S) = k\log k - k\log e + o(k)$.

Proof. First notice that $r = k - O(\log k)$. The number of possibilities for choosing the evaluation of s while keeping ϕ monotonically non-decreasing is:

$$|\text{Solution Space}| = \binom{2^m + r - 1}{r}.$$

Since $2^m \geq k^2 > r^2 = \omega(r)$,

$$\log|\text{Solution Space}| = r\cdot m - \log r! + o(r).$$

On the other hand, $|s| = r\cdot m$. As a result,

$$\text{info}(S) = \log r! + o(r) = r\log r - r\log e + o(r) = k\log k - k\log e + o(k).$$

□

Claim 3.6. When k is sufficiently large, S is fairly assignable.

Proof. The number of edges in s is $r\cdot m = \Theta(k\log k) = o\left(k^2\right)$. For a large enough k, $|s| = o\left(k^2\right) < \Theta\left(k^{d-1}\right) = \binom{k}{d}/2k$, thus $|F_S| > \binom{k}{d}\left(1 - \frac{1}{2k}\right)$, and by Lemma 2.8, S is fairly assignable. □

As a conclusion, S is a fairly assignable system of predicates (for sufficiently large k), describing all d-uniform hypergraphs on k vertices, for $d \geq 3$, and $\mathrm{info}(S) = k \log k - k \log e + o(k)$. $\square$

Combining Theorem 3.2 with Theorem 2.7 we get an explicit construction of a d-uniform k-induced-universal hypergraph whose number of vertices is

$$\begin{aligned} \frac{2 \log e}{e} \cdot k \cdot 2^{\frac{\binom{k}{d} - \mathrm{info}(S)}{k}} &= \frac{2 \log e}{e} \cdot k \cdot 2^{\binom{k}{d}/k} \cdot 2^{-\log k + \log e + o(1)} \\ &= (2 \log e + o(1)) \cdot 2^{\binom{k}{d}/k} \approx (2.89 + o(1))\, 2^{\binom{k}{d}/k}. \end{aligned}$$

4 Tight bound for an induced universal hypergraph

While the results of the previous sections provide constructive ways to generate induced universal hypergraphs, the resulting hypergraphs consist of $c \cdot 2^{\binom{k}{d}/k}$ vertices for some c which is roughly $2.89 > 1$. We now present a non-constructive proof for the existence of a d-uniform k-induced-universal hypergraph on $(1 + o(1))\, 2^{\binom{k}{d}/k}$ vertices. The proof for $d = 2$ was given in [2]. We describe here a proof for $d \geq 3$ using a similar general approach, with several new ingredients, starting with the following definition:

Definition 4.1. A d-uniform hypergraph on k vertices is called *symmetric* if it contains an induced sub-hypergraph on at least $k^{0.85}$ vertices which has a non-trivial automorphism group. A d-uniform hypergraph on k vertices is called *asymmetric* if it is not symmetric.

Our induced universal hypergraph consists of two vertex disjoint parts - an explicit hypergraph on $o\left(2^{\binom{k}{d}/k}\right)$ vertices, containing all symmetric hypergraphs, and a random hypergraph on $(1 + o(1))\, 2^{\binom{k}{d}/k}$ vertices, containing all asymmetric ones. We start with a description of the first part.

Proposition 4.2. *Fix $d \geq 3$. There exists a hypergraph on $o\left(2^{\binom{k}{d}/k}\right)$ vertices, containing all symmetric d-uniform k-hypergraphs.*

Proof. Let G be a symmetric hypergraph. Let K be an induced sub-hypergraph with a non-trivial automorphisms group, where $|V(K)| \geq k^{0.85}$. Let σ be a non-trivial automorphism with minimal order in the automorphism group. Notice that σ is a multiplication of distinct equal-(prime-)length cycles (Otherwise, take a cycle with minimal length p in σ, then $p \mid Ord(\sigma), p \neq Ord(\sigma)$, and σ^p is a non-trivial automorphism in the group, which has a lower order). We now categorize σ as one of several types of permutations. Denote the order of σ by p, the number of its cycles by c, and the number of its fixed elements by s. Let $a_{i,j}$ denote the j-th element of the i-th cycle, and let $b_1, \ldots, b_s$ denote the fixed vertices. Define $m = k^{1.02/(d-1)}$. Clearly, $p \cdot c + s = |V(K)|$.

For each category of permutations, we provide a fairly assignable system of predicates S with $\text{info}(\text{S}) = \Theta\left(k^{1.02}\right)$. Each system S will consist of predicates in the form of equations $e_i = e_j$. To show S is assignable we make sure to use every edge in at most one equation and that each equation has a vertex common to both edges of the equation. The information gained by each equation is 1 bit, and the remaining entropy of its component is 1 bit as well. A fair assignment is given by Lemma 2.8, since $|S| = \Theta\left(k^{1.02}\right) = o\left(k^{d-1}\right)$ and $|F_S| = \binom{k}{d} - o\left(k^{-1}\right)$. Note that it is enough to show a system of $\Omega\left(k^{1.02}\right)$ equations, since it is always possible to remove equations until the system of equations is fairly assignable. Note also that crucially, the equations are independent of the parameters $p, c, s, |V(K)|$ and only depend on k and d.

Case 1:
$s > k^{0.51} \geq k^{1.02/(d-1)}$. Let $a_{1,1}, a_{1,2}$ be two vertices on the same cycle in σ. For every choice of indices $1 \leq i_1 < \ldots < i_{d-1} \leq m < s$, the following equation holds:

$$e_{\{a_{1,1},b_{i_1},\ldots,b_{i_{d-1}}\}} = e_{\{a_{1,2},b_{i_1},\ldots,b_{i_{d-1}}\}}.$$

These add up to $\binom{m}{d-1} = \Omega\left(k^{1.02}\right)$ equations.

Case 2:
$s \leq k^{0.51}, p = 2$. Notice that since $|V(K)| > k^{0.84}$, $c > k^{0.51} \geq m$. For every choice of the indices $1 \leq i_1 < \ldots < i_{d-1} \leq m$, the following equation holds:

$$e_{\{a_{i_1,1},a_{i_1,2},a_{i_2,1},\ldots,a_{i_{d-1},1}\}} = e_{\{a_{i_1,1},a_{i_1,2},a_{i_2,2},\ldots,a_{i_{d-1},2}\}}.$$

These form $\binom{m}{d-1} = \Omega\left(k^{1.02}\right)$ equations, each equation has the common vertices $a_{i_1,1}$ and $a_{i_1,2}$.

Case 3:
$s \leq k^{0.51}, 2 < p < k^{0.34}$. Notice that this implies that $c > k^{0.51} \geq m$. For every choice of the indices $1 \leq i_1 < \ldots < i_{d-1} \leq m$, the following equation holds:

$$e_{\{a_{i_1,1},a_{i_1,2},a_{i_2,1},\ldots,a_{i_{d-1},1}\}} = e_{\{a_{i_1,2},a_{i_1,3},a_{i_2,2},\ldots,a_{i_{d-1},2}\}}.$$

This is similar to case 2, but now the only common vertex is $a_{i_1,2}$.

Case 4:
$s \leq k^{0.51}, k^{1.02/d} \leq k^{0.34} \leq p < k^{0.51}$. Notice that this implies that $c > k^{0.34} \geq k^{1.02/d}$. For every selection of indices $1 \leq i_1 < i_2 < k^{1.02/d}$, indices $1 \leq j_1 = j_2 - 1 < k^{1.02/d}$, where j_1 is even, and indices $1 \leq j_3 < \ldots < j_d < k^{1.02/d}$, the following equation holds:

$$e_{\{a_{i_1,j_1},a_{i_1,j_2},a_{i_2,j_3}\ldots,a_{i_2,j_d}\}} = e_{\{a_{i_1,j_2},a_{i_1,j_2+1},a_{i_2,j_3+1},\ldots,a_{i_2,j_d+1}\}}.$$

These form $\Omega\left(k^{1.02}\right)$ equations, with common vertex a_{i_1,j_2}.

Case 5:
$s \leq k^{0.51}, m \leq k^{0.51} \leq p$. Fix $i = 1 \leq c$. For every selection of indices $1 \leq j_1 = j_2 - 1 < m/2$, where j_1 is even, and distinct indices $m/2 \leq j_3 < \ldots < j_d < m-1$, the following equation holds:

$$e_{\{a_{1,j_1},a_{1,j_2},\ldots,a_{1,j_d}\}} = e_{\{a_{1,j_2},a_{1,j_2+1},a_{1,j_3+1},\ldots,a_{1,j_d+1}\}}.$$

These form $\Omega\left(k^{1.02}\right)$ valid equations, with common vertex a_{1,j_2}.

Using Lemma 2.8 and Theorem 2.7, for each of the five cases we get a hypergraph of size

$$O\left(k\cdot 2^{\frac{\binom{k}{d}-k^{1.02}}{k}}\right)=O\left(k\cdot 2^{\binom{k}{d}/k-k^{0.02}}\right)=o\left(k\cdot 2^{\binom{k}{d}/k-\log k}\right)=o\left(2^{\binom{k}{d}/k}\right).$$

This completes the proof of Proposition 4.2. □

Next, we consider the asymmetric hypergraphs.

Proposition 4.3. *Fix $d\geq 3$. There exists a d-uniform hypergraph with $n=(1+o(1))\,2^{\binom{k}{d}/k}$ vertices containing all d-uniform asymmetric hypergraphs of size k.*

Proof. We start by proving a lemma, limiting the number of ways two copies of an asymmetric hypergraph can intersect.

Lemma 4.4. *Let H be an asymmetric hypergraph of size k, and let K,K' be two sets of labelled vertices of H, each of size k, such that $|K\cap K'|=k-i$, and suppose $k-i>k^{0.85}$.*
The number of ways to set the edges in K and the edges in K' so that H is isomorphic to both the induced sub-hypergraph on K and the induced sub-hypergraph on K' is at most $k!\cdot k^i$.

Proof. Since H has no non-trivial automorphisms, there are $k!$ ways to map $V(H)$ to K. Given that embedding, there are now at most $k\,(k-1)\ldots(k-i+1)<k^i$ ways to choose which vertices of H are mapped to $K'-K$. Denote them by T. Since the induced sub-hypergraph of H on $V(H)-T$ has $k-i>k^{0.85}$ vertices, it has no non-trivial automorphisms, so there is at most one way to embed the vertices of $V(H)-T$ to $K-K'$ in a way consistent with the existing edges. □

In addition, we use a known theorem by Talagrand, see e.g. [1], Chapter 7.

Theorem 4.5 (Talagrand's Inequality)**.** *Let $\Omega=\prod_{i=1}^{p}\Omega_i$, where each Ω_i is a probability space and Ω has the product measure, and let $h:\Omega\to\mathbb{R}$ be a function. Assume that h is Lipschitz, that is, $|h(x)-h(y)|\leq 1$ whenever x,y differ in at most one coordinate. For a function $f:\mathbb{N}\to\mathbb{N}$, h is f-*certifiable *if whenever $h(x)\geq s$ there exists $I\subseteq\{1,\ldots,p\}$ with $|I|\leq f(s)$ so that for every $y\in\Omega$ that agrees with x on the coordinates I we have $h(y)\geq s$. Suppose that h is f-*certifiable *and let Y be the random variable given by $Y(x)=h(x)$ for $x\in\Omega$. Then for every b and t*

$$Prob\left[Y\leq b-t\sqrt{f(b)}\right]\cdot Prob\left[Y\geq b\right]\leq e^{-t^2/4}.$$

Let n to be the minimum integer such that $\binom{n}{k}k!2^{-\binom{k}{d}} \geq k^{2d}$. Put $\mu = \binom{n}{k}k!2^{-\binom{k}{d}}$. It is easy to check that $\mu = (1+o(1))\,k^{2d}$ and $n = (1+o(1))\,2^{\binom{k}{d}/k}$.

Let $G \sim G(n,d,0.5)$ be the random d-uniform hypergraph on n vertices. Let H be a fixed d-uniform asymmetric hypergraph on k vertices. Since H is asymmetric, it has no non-trivial automorphisms, so $|Aut(H)| = 1$. For a subset $K \subset V(G)$ of k vertices, let X_K be the random variable indicating whether H is isomorphic to the induced sub-hypergraph of G on K.

$$E(X_K) = P(X_K = 1) = \frac{k!}{|Aut(H)|}2^{-\binom{k}{d}} = k! \cdot 2^{-\binom{k}{d}}.$$

Let X be the random variable counting the number of appearances of H in G, that is:

$$X = \sum_{K \subset V(G), |K|=k} X_K$$

$$E(X) = \binom{n}{k}k!2^{-\binom{k}{d}} = \mu.$$

Denote by $K \sim K'$ the fact that $d \leq |K \cap K'| < k$. Let Z denote the number of ordered pairs of intersecting copies of H in G, that is:

$$Z = \sum_{K \sim K' \subset V(G)} X_K X_{K'}.$$

(Note that if two sets of vertices share less than d common vertices, then they have no common edges or non-edges). Denote by Δ_j the expected number of ordered pairs of intersecting copies with intersection of size exactly j, that is:

$$\Delta_j = E\left(\sum_{|K \cap K'|=j} X_K X_{K'}\right)$$

and let Δ denote the total expected number of intersecting pairs, i.e. $\Delta = E(Z) = \sum_{j=d}^{k-1}\Delta_j$. We next provide an upper bound to Δ/μ.

Case 1: $d \leq j \leq k^{0.85}$.

$$\Delta_j \leq \binom{n}{k}k!\binom{n-k}{k-j}\binom{k}{j}k!2^{-2\binom{k}{d}+\binom{j}{d}}.$$

Indeed, this is a naive upper bound for the number of ways to choose the vertices and the embeddings, and the probability that the induced sub-hypergraphs of G are isomorphic to H. Since $j \leq k^{0.85} < \frac{k}{2}$, $\frac{\binom{j}{d}}{j} < \frac{\binom{k}{d}}{k2^{d-1}}$, and therefore $2^{\binom{j}{d}} < n^{0.25j}$. Therefore:

$$\frac{\Delta_j}{\mu^2} \leq \frac{\binom{n-k}{k-j}\binom{k}{j}2^{\binom{j}{d}}}{\binom{n}{k}} < n^{-j}k^{2j}n^{0.25j} = \left(\frac{k^2}{n^{0.75}}\right)^j < \left(\frac{1}{n^{0.74}}\right)^j < n^{-2}.$$

Since $\mu < n$, $\frac{\Delta_j}{\mu} < n^{-1}$.

Case 2: $k - i = j > k^{0.85}$. By Lemma 4.4

$$\Delta_j \leq \binom{n}{k}\binom{n-k}{k-j}\binom{k}{j}k!k^i 2^{-2\binom{k}{d}+\binom{j}{d}}.$$

Therefore:

$$\frac{\Delta_j}{\mu} \leq \binom{n-k}{i}\binom{k}{i}k^i 2^{-\binom{k}{d}+\binom{j}{d}}.$$

Note that

$$\begin{aligned}
\binom{k}{d} - \binom{j}{d} &= \binom{k}{d} - \binom{k-i}{d} \\
&= \frac{k(k-1)\dots(k-d+1) - (k-i)\dots(k-i-d+1)}{d!} \\
&\geq \frac{k(k-1)\dots(k-d+1) - (k-1)\dots(k-d+1)(k-i-d+1)}{d!} \\
&= \binom{k}{d} \cdot (i+d-1)/k.
\end{aligned}$$

Thus:

$$\frac{\Delta_j}{\mu} < n^i k^{2i} 2^{-\binom{k}{d}/k\cdot(i+d-1)} < n^i k^{2i} n^{-i-d+1+o(1)} < \frac{k^{2k}}{n^{d-1-o(1)}} < n^{-1}.$$

Overall, we get that $\Delta < \sum_{j=d}^{k-1} n^{-1}\mu < \frac{k\mu}{n} = o(\mu)$. Notice that

$$\begin{aligned}
Var(X) &\leq E(X) + \sum_{K\sim K'} Cov(X_K, X_{K'}) \\
&\leq E(X) + \sum_{K\sim K'} (E(X_K X_{K'}) - E(X_K)E(X_{K'})) \\
&\leq E(X) + \sum_{K\sim K'} E(X_K X_{K'}) = E(X) + \Delta = \mu + o(\mu).
\end{aligned}$$

By Chebychev's inequality, with high probability, $X \geq \frac{3\mu}{4}$, and by Markov's inequality, with high probability, $Z < \frac{\mu}{4}$. Both events happen simultaneously with high probability. By excluding a copy of H from every pair of intersecting copies, we get that with probability greater than $1/2$, there are at least $\mu/2$ pairwise edge-disjoint copies of H.

For the random hypergraph G, let $Y = Y(H, G)$ be the maximal number of pairwise edge-disjoint copies of H in G, then $Y \geq \mu/2$ with probability greater than $1/2$. By modifying a single edge in G, the value of Y can change by at most 1, and Y is f-certifiable where $f(s) = s\binom{k}{d}$.

Using Talagrand's theorem, with $b = \mu/d!$ and $t = \sqrt{\mu}k^{-d/2}$, we get

$$P(Y=0)P(Y \geq b) \leq e^{-\mu k^{-d}/4} \leq e^{-k^d/4}.$$

Furthermore, $P(Y \geq b) > 0.5$. We conclude that $P(Y = 0) < 2^{-\binom{k}{d}}$. As the number of d-uniform hypergraphs on k-vertices is at most $2^{\binom{k}{d}}$, it follows that with positive probability (and in fact, with high probability), all asymmetric hypergraphs appear a positive number of times in G. □

Theorem 4.6. *For $d \geq 3$, there exists a d-uniform k-induced-universal hypergraph whose number of vertices is*

$$n = (1 + o(1))\, 2^{\binom{k}{d}/k}.$$

Proof. Combining the results from Proposition 4.2 and Proposition 4.3, we get a single induced universal hypergraph, containing all symmetric and asymmetric d-uniform hypergraphs, with the required size. □

5 Concluding remark and open problems

We have shown that the minimum possible number of vertices in a d-uniform k-induced-universal hypergraph is $(1 + o(1))\, 2^{\binom{k}{d}/k}$. The proof does not provide an explicit construction of such a hypergraph and it will be interesting to find one. Note that as proved in Claim 3.1, our methods for obtaining explicit hypergraphs described in Sections 2 and 3 cannot be used to obtain an induced universal hypergraph with less than $(e + o(1))\, 2^{\binom{k}{d}/k}$ vertices. An equivalent way to see the limitation of the method is the fact that our construction provides a hypergraph in which the vertex set is partitioned into k disjoint parts and each hypergraph on k vertices appears as an induced hypergraph containing exactly one vertex from each part. Therefore, if n is the number of vertices of the constructed hypergraph, then $\left(\frac{n}{k}\right)^k \geq \frac{2^{\binom{k}{d}}}{k!}$, implying that $n \geq (e + o(1))\, 2^{\binom{k}{d}/k}$.

We note that in our explicit construction for d-uniform k-induced-universal hypergraphs where $d \geq 3$, the number of vertices is rather close to $e \cdot 2^{\binom{k}{d}/k}$. For $d = 2$, that is, the case of graphs, the best known explicit construction has roughly $20 \cdot 2^{\binom{k}{d}/k}$ vertices, see [4],[14]. It may be interesting to improve it.

It may also be interesting to improve the gap between the upper and lower bound for the minimum possible number of vertices in a (not necessarily explicit) d-uniform k-induced-universal hypergraph, although the gap is negligible with respect to the main term. The only lower bound we have comes from simple counting and it will be interesting to improve it.

Other questions that may be considered are the estimation of the minimum possible size of induced universal hypergraphs for other families including, possibly, non uniform ones. Some of our techniques here can be used in many cases.

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