Graph Theory
Homework assignment #1
Due date: Sunday, November 15, 2015

Problem 1. Prove that for each \( n \geq 1 \), the number of graphs with vertex set \( \{1, \ldots, n\} \) and all degrees even is \( 2^{\binom{n-1}{2}} \).

Problem 2. Suppose that \( n \geq 8 \). Prove that every \( n \)-vertex graph graph with at least \( 6n - 20 \) edges contains a subgraph with minimum degree at least 7.

Problem 3. Let \( G \) be a graph with \( n \) vertices. Prove that \( G \) contains a cycle with a chord (an edge connecting nonconsecutive vertices of the cycle) if either

(a) \( \delta(G) \geq 3 \) or

(b) \( |E(G)| \geq 2n - 3 \) and \( n \geq 4 \).

Problem 4. Prove that every graph \( G \) with \( m \) edges admits a bipartition \( V(G) = V_1 \cup V_2 \) such that the number of edges of \( G \) crossing between \( V_1 \) and \( V_2 \) is at least \( m/2 \).

Problem 5. Let \( d_1, \ldots, d_n \) be positive integers. Prove that there exists a tree with degrees \( d_1, \ldots, d_n \) if and only if

\[
d_1 + \ldots + d_n = 2n - 2.
\]

Problem 6. Prove that if \( T_1, \ldots, T_k \) are pairwise intersecting subtrees of a tree \( T \), then \( T \) has a vertex that belongs to each of \( T_1, \ldots, T_k \).

Problem 7. Prove that every graph \( G \) contains each tree with \( \delta(G) \) edges as a subgraph.

Problem 8. Compute the number of spanning trees of the complete bipartite graph \( K_{m,n} \).

Please do NOT submit written solutions to the following exercises:

Exercise 1. Show that a graph is bipartite if and only if it contains no odd cycles. In particular, all trees are bipartite.

Exercise 2. Suppose that \( m \leq n \), let \( A \) be an \( m \times n \) matrix and let \( B \) be an \( n \times n \) matrix. Prove, using the Lindström–Gessel–Viennot lemma, the Cauchy–Binet formula:

\[
det AB = \sum_{J \subseteq \binom{[n]}{m}} det A_J \cdot det B_J,
\]

where \( A_J \) is the \( m \times m \) submatrix of \( A \) consisting of the columns indexed by \( J \) and \( B_J \) is the \( m \times m \) submatrix of \( B \) consisting of the rows indexed by \( J \).

Exercise 3. Show that the block graph of a connected graph is a tree.