Probabilistic Methods in Combinatorics: Homework Assignment Number 2

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Solutions will be collected in class on Monday, December 3, 2012.

1. For each i, $1 \leq i \leq m = n^{n^2+1}$ let
$$\{A_1^{(i)}, A_2^{(i)}, \ldots, A_n^{(i)}\}$$
be a collection of n pairwise disjoint subsets of the set of integers Z, where $|A_j^{(i)}| = n$ for all i and j. Prove that there are n distinct numbers $1 \leq i_1 < i_2 < \ldots < i_n \leq m$ so that the n sets
$$\{A_1^{(i_1)}, A_2^{(i_2)}, \ldots, A_n^{(i_n)}\}$$
are pairwise disjoint.

2. Show that there are infinitely many values of n so that for more than 0.9 of the graphs G on n labelled vertices the size of the largest clique in G is equal exactly to the size of the largest independent set in it.

3. Let $v_1 = (x_1, y_1), \ldots, v_n = (x_n, y_n)$ be n two dimensional vectors, where each x_i and each y_i is a positive integer that does not exceed $\frac{2^{n/2}}{10\sqrt{n}}$. Show that there are two disjoint nonempty sets $I, J \subset \{1, 2, \ldots, n\}$ such that
$$\sum_{i \in I} v_i = \sum_{j \in J} v_j.$$

4. Let $A = (a_{ij})$ be an n by n matrix where $a_{ij} \in \{0, 1\}$ for all i, j, and suppose further that every column of A contains exactly $n/2$ ones (and hence also exactly $n/2$ zeros). Let A' be a random matrix obtained from A by applying a random permutation to each column of A, where all n permutations are chosen independently and uniformly. Prove that if n is large enough then with probability at least 0.9999 the number of rows of the permuted matrix A' in which the number of ones is between $n/2 - \sqrt{n}$ and $n/2 + \sqrt{n}$ is at least $n/4$.

5. (i). Show that for any two integers k and ℓ and for any real p, $0 < p < 1$, and any integer n, the Ramsey number $r(k, \ell)$ is at least
$$n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{\ell} (1-p)^{\binom{\ell}{2}}.$$
(ii). Apply the above to prove that the Ramsey number $r(4, k)$ satisfies $r(4, k) \geq c(k/\ln k)^{\alpha}$ for some absolute constant $c > 0$ and for the largest $\alpha > 0$ for which you can derive this inequality from the result in (i).

6. For a tournament $T = (V, E)$ and for two disjoint subsets A, B of V, let $e(A, B)$ denote the number of edges of T directed from a vertex of A to a vertex of B, that is

$$e(A, B) = |\{ \ (x, y) \in E : x \in A, y \in B\}|.$$

(i) Prove that there exists an absolute constant $c > 0$ so that for any tournament $T = (V, E)$ on n vertices there are two disjoint subsets A and B of V such that $e(A, B) - e(B, A) \geq cn^{3/2}$.

(ii) Prove that there exists an absolute constant $b > 0$ so that for any tournament $T = (V, E)$ on n vertices there is a linear order L of the vertices so that the number of directed edges (i, j) of T for which i precedes j in L is at least

$$\frac{1}{2}\binom{n}{2} + bn^{3/2}.$$