Probabilistic Methods in Combinatorics: Homework Assignment Number 3
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Solutions will be collected in class on Monday, December 24, 2012.

1. A simple path of an even length $P = v_1 v_2 \cdots v_{2k}$ in a graph $G = (V, E)$ with a vertex coloring $f : V \mapsto Z$ is *periodic* if $f(v_j) = f(v_{k+j})$ for all j, $1 \leq j \leq k$. Prove that there is a finite r so that the following holds. For every graph $G = (V, E)$ with maximum degree 5 and for every assignment of sets $S_v \subset Z$ of colors for each $v \in V$, where $|S_v| = r$ for all v, there is a vertex coloring f assigning to each vertex v a color $f(v) \in S_v$ such that no simple path (of any even length) is periodic.

2. Prove that there is an absolute constant $c > 0$ so that for every $d > 2$, every d-regular graph contains a spanning subgraph consisting of vertex disjoint stars, each having at least $cd/\log d$ edges.

3. Is the following statement correct ? (prove or supply a counter-example).

   There exists an absolute constant $b > 0$ so that for every $d > 2$, every d-regular graph contains a spanning subgraph consisting of vertex disjoint stars, each having at least bd edges.

4. Let $H = (V, E)$ be a graph containing r triangles, and let S be a random set of vertices of H obtained by picking each $v \in V$ randomly and independently, with probability 1/2. Prove that the probability that S contains no triangle of H is at least $(7/8)^r$.

5. Let w be a random vector of length n with $\{0, 1\}$ entries obtained by selecting each entry, randomly and independently, to be 1 with probability 1/3, and 0 with probability 2/3. Let $v_1, v_2, \ldots, v_k$ be k distribution vectors, each of length n (that is, each v_i has n non-negative coordinates whose sum is 1). Show that the probability that the inner product of w and v_i is at least 1/3 for each i, $1 \leq i \leq k$, is at least $(1/3)^k$.