Probabilistic Methods in Combinatorics: Homework Assignment Number 4
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Solutions will be collected in class on Monday, January 14, 2013.

1. Prove that for any $\epsilon > 0$ there are $c = c(\epsilon)$ and $n_0 = n_0(\epsilon)$ so that for any $n > n_0$ there is an interval I_n of at most $c\frac{\sqrt{n}}{\log n}$ integers such that the probability that the chromatic number of the random graph $G(n, 1/2)$ lies in I_n is at least $1 - \epsilon$.

2. Prove that for any positive integer $k > 1$ there is a $c = c(k)$ so that for any collection of subsets $A_1, A_2, \ldots, A_k \subset \{0,1\}^n$ that satisfy $|A_i| \geq \frac{2^n}{k}$ for all i, there are points $v_i \in A_i$ such that any pair of the points v_i, v_j $(1 \leq i < j \leq k)$ differ in at most $c\sqrt{n}$ coordinates.

3. Consider any permutation $f \in S_n$ as a (bijective) function from $[n] = \{1, 2, \ldots, n\}$ to $[n]$. The distance between two permutations f_1, f_2 is $d(f_1, f_2) = |\{i : 1 \leq i \leq n, f_1(i) \neq f_2(i)\}|$.

   (i) Prove that for any $\epsilon > 0$ there is a $c = c(\epsilon)$ so that for any two sets F_1, F_2 each consisting of at least $\epsilon n!$ permutations in S_n, there is an $f_1 \in F_1$ and an $f_2 \in F_2$ so that

$$d(f_1, f_2) \leq c\sqrt{n}. \qquad (1)$$

   (ii) Is the statement obtained from that in (i) by replacing (1) by

$$d(f_1, f_2) \leq cn^{1/3}$$

   correct ?

4. Prove that there exists a positive constant $\delta > 0$ and an integer $n_0 = n_0(\delta)$ so that for all $n > n_0$ and every collection $\{(S_1, T_1), (S_2, T_2), \ldots, (S_m, T_m)\}$, where $m \leq 2^{\delta n}$, of pairs of subsets of $[4n] = \{1, 2, \ldots, 4n\}$, satisfying $|S_i| = |T_i|$ for all i, there is a function $f : [4n] \mapsto [n] = \{1, 2, \ldots, n\}$ so that $|f^{-1}(j)| = 4$ for each $j \in [n]$ and for every i, $1 \leq i \leq m$,

$$|\ |f(S_i)| - |f(T_i)|\ | \leq 0.001n.$$

5. Using Janson's Inequality, find a threshold function for the property: $G(n, p)$ contains at least $n/10$ pairwise vertex disjoint copies of K_5.