Probabilistic Methods in Combinatorics: Homework Assignment Number 3

Noga Alon

Solutions will be collected in class on Tuesday, May 23, 2017.

1. Prove that there is an absolute constant $C > 0$ so that for every integer $k \geq 1$, every $\lceil Ck \rceil$-regular directed graph (that is, a graph in which all indegrees and all outdegrees are exactly $\lceil Ck \rceil$) contains k pairwise vertex disjoint even directed cycles (that is, cycles of even lengths).

2. Consider the covering of the plane by the grid squares, that is, by the unit squares whose vertices are the grid points. Prove that there is an absolute constant $C > 0$ so that for every $0.1 > \epsilon > 0$ there is a set F of points in the plane containing exactly one point in each grid square so that every planar (not necessarily aligned) rectangle of dimensions ϵ by $C\frac{\log(1/\epsilon)}{\epsilon}$ contains at least one point of F.

3. Prove that there is an absolute constant $C > 0$ so that for every (finite or infinite) group H, every integer $k \geq 1$ and every subset $A \subset H$ of size $|A| = \lceil Ck \log k \rceil$ there is a function $f : H \mapsto [k] = \{1, 2, \ldots, k\}$ so that for every $h \in H$, $f(hA) = [k]$.

4. Let w be a random vector of length n with $\{0, 1\}$ entries obtained by selecting each entry, randomly and independently, to be 1 with probability $1/3$, and 0 with probability $2/3$. Let $v_1, v_2, \ldots, v_k$ be k distribution vectors, each of length n (that is, each v_i has n non-negative coordinates whose sum is 1). Show that the probability that the inner product of w and v_i is at least $1/3$ for each i, $1 \leq i \leq k$, is at least $(1/3)^k$.

5. Is the following statement correct ? (prove or supply a counterexample):

   Let $G_1, G_2, \ldots, G_t$ be a collection of $t = n^{20}$ graphs on the same set of vertices V, $|V| = n$. Suppose the chromatic number of each G_i is at least $r = 3n^{0.1}$, and suppose that n is sufficiently large. Then there is a subset $U \subset V$ of size at most $2n/3$, so that for every i, $1 \leq i \leq t$, the chromatic number of the induced subgraph of G_i on U is at least $\frac{r}{3} = n^{0.1}$.

6. Is the following statement correct ? (prove or supply a counterexample):

   Let $G_1, G_2, \ldots, G_t$ be a collection of $t = n^{20}$ graphs on the same set of vertices V, $|V| = n$. Suppose the chromatic number of each G_i is at least $r = 3\log_2 n$, and suppose that n is sufficiently large. Then there is a subset $U \subset V$ of size at most $2n/3$, so that for every i, $1 \leq i \leq t$, the chromatic number of the induced subgraph of G_i on U is at least $\frac{r}{3} = \log_2 n$.