Localization of Multi-Dimensional Wigner Distributions

Elliott H. Lieb and Yaron Ostrover

August 10, 2010 *

Abstract

A well known result of P. Flandrin states that a Gaussian uniquely maximizes the integral of the Wigner distribution over every centered disc in the phase plane. While there is no difficulty in generalizing this result to higher-dimensional poly-discs, the generalization to balls is less obvious. In this note we provide such a generalization.

1 Introduction

The Wigner quasi-probability distribution was introduced by Wigner [16] in 1932 in order to study quantum corrections to classical statistical mechanics. Nowadays it lies at the core of the phase-space formulation of quantum mechanics (Weyl correspondence), and has a variety of applications in statistical mechanics, quantum optics, and signal analysis, to name a few. In this note we consider the localization problem of the $n$-particle Wigner distribution in the $2n$-dimensional phase space. We state our results precisely in Theorem 1 below.

Equip the classical phase space $\mathbb{R}^{2n}$ with coordinates $(x, y)$ with $x, y \in \mathbb{R}^n$. The Wigner quasi-probability distribution on $\mathbb{R}^{2n}$, associated with a wave function $\psi \in L^2(\mathbb{R}^n)$ and its complex conjugate $\psi^\ast$, is defined by

$$W_\psi(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} \psi(x + \tau/2)\psi^\ast(x - \tau/2)e^{-i\tau \cdot y} d\tau$$

The function $W_\psi$ possesses many of the properties of a phase space probability distribution (see e.g., [4]); in particular, it is real. However, $W_\psi$ is not a genuine probability distribution as it can assume negative values.

The localization problem, i.e., estimating the integral of the Wigner distribution over a subregion of the phase space, and the closely related problem of the optimal simultaneous concentration of $\psi$ and its Fourier transform $\hat{\psi}$, have received much attention in the literature both in quantum mechanics, mathematical time-frequency analysis, and signal
processing (see e.g. [1, 2, 3, 4, 5, 6, 9, 10, 12, 13, 11], and the references within). Bounds on the $L^p$ norms were found in [7]. More precisely, the problem of interest for us is:

**The Wigner Distribution Localization Problem:** given a measurable set $D \subset \mathbb{R}^{2n}$, find the best possible bounds to the localization function

$$E(D) := \sup_{\psi} \int_D W_\psi \, dx \, dy,$$

where the supremum is taken over all the functions $\psi \in L^2(\mathbb{R}^n)$ with $\|\psi\|_2 = 1$.

The quantity $E(D)$ is invariant under translations in the phase space, and under the action of the group of linear symplectic transformations (see e.g. [15]). There is no upper bound on $E(D)$; it can be infinite. Indeed, there is a $\psi \in L^2(\mathbb{R}^n)$ such that $\int |W_\psi| \, dx \, dy = \infty$ [4, sect. 4.6]. An example is $\psi(x) = 1$ if $-\frac{1}{2} < x < \frac{1}{2}$ and $\psi(x) = 0$ otherwise. On the other hand, the $L^p$ norm of $W_\psi$ is bounded [7] for $p \geq 2$ and we can use this information to show that $E(D)$ is bounded by powers of the volume $|D|$. E.g., the $L^\infty$ norm is at most $\pi^{-n}$, so $E(D) \leq \pi^{-n} |D|$.

For certain $D$, however, $E(D)$ is not only finite, it is even less than 1. In [2], Flandrin conjectured this to be true for all convex domains, and he showed that for all centered two-dimensional discs $B^2(r)$ of radius $r$, the standard normalized Gaussian function $\pi^{-1/4} \exp(-x^2/2)$ is the unique maximizer of (1.2). In particular $E(B^2(r)) = 1 - e^{-r^2}$ (see [2], cf. [4]). It follows immediately from the definition of the Wigner distribution that Flandrin’s proof can be easily generalized to higher dimensional poly-discs because the maximization problem then has a simple product structure. A less obvious case is the $2n$-dimensional Euclidean ball $B^{2n}(r)$. The following is the generalization of Flandrin’s result, and our main result:

**Theorem 1.** The standard normalized Gaussian $\pi^{-n/4} \exp(-x^2/2)$ in $L^2(\mathbb{R}^n)$ is the unique maximizer of the Wigner distribution localization problem for any $2n$-dimensional Euclidean ball centered at the origin. In particular,

$$E(B^{2n}(r)) = \frac{1}{\pi^n} \int_{B^{2n}(r)} e^{-\sum_{i=1}^{n} (x_i^2+y_i^2)} \, dx \, dy = 1 - \frac{\Gamma(n, r^2)}{(n-1)!},$$

where $\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} \, dt$ is the upper incomplete gamma function.

Remarks: (1.) Owing to the translation covariance of the Wigner distribution, equation (1.3) also applies to a ball of radius $r$ centered anywhere in $\mathbb{R}^{2n}$. It is only necessary to multiply the Gaussian by an appropriate linear form $\exp(a \cdot x)$. Moreover, since the localization function (1.2) is invariant under the action of the group of linear symplectic transformations, Theorem 1 can also be adapted to any image of the Euclidean ball under linear symplectic maps.

(2.) Another generalization is to replace the integral over the ball with the integral over $\mathbb{R}^{2n}$, but with a weight that is a symmetric decreasing function (i.e., a radial and non-increasing function of the radius $\sqrt{x^2+y^2}$). By the “layer cake representation” [8, sect. 1.13] the standard Gaussian again maximizes uniquely.
2 Proof of Theorem 1

We start with the following preliminaries. Recall that the mixed Wigner distribution of two states $\psi_1, \psi_2 \in L^2(\mathbb{R}^n)$ is defined by

$$W_{\psi_1,\psi_2}(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} \psi_1(x+\tau/2)\psi_2^*(x-\tau/2)e^{-i\tau y} \, d\tau.$$  \hspace{1cm} (2.1)

Note that in contrast to (1.1), $W_{\psi_1,\psi_2}$ is not generally real, but, nevertheless, Hermitian i.e., $W_{\psi_1,\psi_2} = W_{\psi_2,\psi_1}^*.$ Moreover, it is not hard to check that the mixed Wigner distribution is sesquilinear.

Next, let $\mu = (\mu_1, \ldots, \mu_n)$ be a multi-index of non-negative integers, and let $x \in \mathbb{R}^n.$ The Hermite functions $H_\mu(x)$ on $\mathbb{R}^n$ are defined [14, 15] to be the product of the normalized one-dimensional Hermite functions, i.e., $H_\mu(x) = \prod_{j=1}^n h_{\mu_j}(x_j),$ where

$$h_k(x) = \pi^{-\frac{k}{4}} (k!)^{-\frac{1}{2}} 2^{-\frac{k}{2}} (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2}.$$  \hspace{1cm} (2.2)

It is well known that the $\{H_\mu\}$ form a complete orthonormal system for $L^2(\mathbb{R}^n),$ and that

$$\mathbb{H} H_\mu = |\mu| H_\mu,$$  \hspace{1cm} (2.3)

where $|\mu| = \sum_{j=1}^n \mu_j,$ and $\mathbb{H}$ is the Schrödinger operator $\mathbb{H} = -\frac{1}{2}\Delta + \frac{1}{2}|x|^2 - \frac{x_1}{2}.$ Here $\Delta$ denotes the standard $n$-dimensional Laplacian. In particular, the sesquilinearity of the Wigner distribution implies that for any $\psi \in L^2(\mathbb{R}^n),$ one has

$$W_\psi = \sum_{\mu} \sum_{\nu} \langle \psi, H_\mu \rangle \langle \psi, H_\nu \rangle^* W_{H_\mu, H_\nu}.$$  \hspace{1cm} (2.4)

The following lemma shows that the integral of the off-diagonal elements of (2.4) over any centered ball $B_{2^n}(r)$ vanishes (cf. [5] Section 2.3).

Lemma 2.1. Let $\mu, \nu$ be two multi-indices with $\mu \neq \nu.$ Then, for every $r \geq 0,$ one has

$$\int_{B_{2^n}(r)} W_{H_\mu, H_\nu} \, dx \, dy = 0.$$  \hspace{1cm} (2.5)

Proof of Lemma 2.1. It is well known (see e.g. [6]) that for the one-dimensional Hermite functions $\{h_m\},$ one has:

$$W_{h_j, h_k}(x_1, y_1) = \begin{cases} \pi^{-1} (k!/j!)^{1/2} (-1)^k (\sqrt{2}\pi)^{j-k} e^{-|z_1|^2} L_k^{j-k}(2|z_1|^2) & \text{if } j \geq k, \\ \pi^{-1} (j!/k!)^{1/2} (-1)^j (\sqrt{2}\pi)^{k-j} e^{-|z_1|^2} L_j^{k-j}(2|z_1|^2) & \text{if } k \geq j. \end{cases}$$  \hspace{1cm} (2.6)

Here $z_1 = x_1 + iy_1,$ and $L_n^\alpha$ are the Laguerre polynomials defined by

$$L_j^\alpha(x) = \frac{x^{-\alpha} e^x}{j!} \frac{d^j}{dx^j} (e^{-x} x^{j+\alpha}),$$  \hspace{1cm} (2.7)

for $j \geq 0$ and $\alpha > -1.$ Hence the lemma holds in the 2-dimensional case, i.e., when $n = 1,$ because the integral of $z^j$ or $\bar{z}^j$ over any circle centered at the origin equals zero when
\( j \neq 0 \). The higher-dimensional case follows for the same reason from (2.6), the fact that the Wigner distribution function \( W_{H_\mu,H_\nu}(x,y) \) is the product of \( W_{h_{m_j},h_{n_j}}(x_j,y_j) \), and the rotation invariance of the ball \( B^{2n}(r) \).

An immediate corollary of Lemma 2.1, definition (1.2), and equality (2.4) is

**Corollary 2.2.** In the notation above,

\[
\mathcal{E}(B^{2n}(r)) = \sup_{\mu} \int_{B^{2n}(r)} W_{H_\mu} dxdy, \tag{2.8}
\]

where the supremum is taken over all multi-indices \( \mu = (\mu_1, \ldots, \mu_n) \) of non-negative integers.

The following lemma is the main ingredient in the proof of Theorem 1.

**Lemma 2.3.** For any integer \( \lambda \geq 0 \) and multi-indices \( \mu_1, \mu_2 \) with \( \lambda = |\mu_1| = |\mu_2| \), one has

\[
\int_{B^{2n}(r)} W_{H_{\mu_1}} dxdy = \int_{B^{2n}(r)} W_{H_{\mu_2}} dxdy, \quad \text{for every } r \geq 0. \tag{2.9}
\]

Postponing the proof of Lemma 2.3, we first conclude the proof of Theorem 1.

**Proof of Theorem 1.** It follows from Corollary 2.2 and Lemma 2.3 above that

\[
\mathcal{E}(B^{2n}(r)) = \sup_{\lambda} \int_{B^{2n}(r)} W_{H_{\mu_\lambda}} dxdy, \tag{2.10}
\]

where \( \mu_\lambda = (\lambda,0,\ldots,0) \), and \( \lambda \) is a non-negative integer. Moreover, from (2.6) and the definition of the Wigner distribution it follow that:

\[
W_{H_{\mu_\lambda}}(x,y) = \frac{(-1)^\lambda}{\pi^n} e^{-\sum_{i=1}^n (x_i^2 + y_i^2)} L_\lambda(2(x_1^2 + y_1^2)), \tag{2.11}
\]

where \( L_\lambda(z) \) are the \( \alpha = 0 \) Laguerre polynomials (2.7). Setting \( z_j = x_j + iy_j \), we conclude that

\[
\int_{B^{2n}(r)} W_{H_{\mu_\lambda}} dxdy = \int e^{-\sum_{j=2}^n |z_j|^2} \left( \int_{|z_1|^2 \leq r^2 - \sum_{j=2}^n |z_j|^2} \frac{(-1)^\lambda}{\pi^n} e^{-|z_1|^2} L_\lambda(2|z_1|^2) d\Gamma(z_1) \right) dz_2 \cdots dz_n. \tag{2.12}
\]

On the other hand, from Flandrin’s result in the 1-dimensional case [2], it follows that

\[
\int_{B^2(\alpha)} W_{h_\lambda} dx_1 dy_1 = \int \frac{(-1)^\lambda}{\pi^n} e^{-|z_1|^2} L_\lambda(2|z_1|^2) d\Gamma(z_1) \leq \int e^{-|z_1|^2} d\Gamma(z_1), \tag{2.13}
\]

for every radius \( \alpha \geq 0 \). An examination of Flandrin’s proof reveals that the inequality is strict for \( \lambda > 0 \). Hence, for every non-negative integer \( \lambda \) one has

\[
\int_{B^{2n}(r)} W_{H_{\mu_\lambda}} dxdy \leq \pi^{-n} \int e^{-\sum_{j=1}^n |z_j|^2} d\Gamma(z_1) \cdots d\Gamma(z_n) = 1 - \frac{\Gamma(n,r^2)}{(n-1)!} \tag{2.14}
\]

with equality only for \( \lambda = 0 \). The proof of Theorem 1 now follows from (2.11) and (2.14). \( \square \)
Remark: The integral in (2.10) is not monotone in \( \lambda \) or in \( r \) (except for \( \lambda = 0 \)), as might have been thought. See [1, Fig. 2] and [2] for interesting graphs of these integrals as a function of \( r \).

For the proof of Lemma 2.3 we shall need the following preliminaries. For a non-negative integer \( \lambda \) denote

\[
\mathcal{H}_\lambda = \text{span}\{H_\mu : |\mu| = \lambda\} \subset L^2(\mathbb{R}^n). \tag{2.15}
\]

It follows from (2.3) above that the space \( \mathcal{H}_\lambda \) consists of the eigenfunctions of the rotation invariant Schrödinger operator \( \mathcal{H} = -\frac{1}{2}\Delta + \frac{1}{2}|x|^2 - \frac{n}{2} \) with eigenvalue \( \lambda \). In particular, it is a finite-dimensional, \( O(n) \)-invariant subspace of \( L^2(\mathbb{R}^n) \) with orthonormal basis \( \{H_\mu : |\mu| = \lambda\} \). It follows that for every \( R \in O(n) \), and every \( \tilde{\mu} \) with \( |\tilde{\mu}| = \lambda \), one has:

\[
H_\mu(Rx) = \sum_{\nu : |\nu| = \lambda} c_\nu(\tilde{\mu}, \mathcal{R}) H_\nu(x), \tag{2.16}
\]

where the coefficients \( c_\nu(\tilde{\mu}, \mathcal{R}) \) satisfy \( \sum |c_\nu(\tilde{\mu}, \mathcal{R})|^2 = 1 \).

We note the following useful fact: In order to identify which coefficients \( c_\nu(\tilde{\mu}, \mathcal{R}) \) are non-zero, it is only necessary to check the leading powers on the two sides of (2.16). That is, the left side of (2.16) defines a polynomial of degree \( \lambda \) in the indeterminates \( x_1, \ldots, x_n \). The highest degree terms are the monomials \( x_1^{\mu_1} \cdots x_n^{\mu_n} \) with \( \sum_{j=1}^n \mu_j = \lambda \), but there are also monomials of degree lower than \( \lambda \). In order to show that a given \( H_\mu \) appears with a non-zero coefficient in the decomposition (2.16), it is only necessary to show that there is a highest degree monomial \( x_1^{\nu_1} \cdots x_n^{\nu_n} \) in the decomposition. It is not necessary to check the lower degree monomials; they will appear automatically because we know that the decomposition contains only Hermite functions of degree \( \lambda \) and no others.

Proof of Lemma 2.3: Fix a non-negative integer \( \lambda \), and \( r \geq 0 \). We consider the maximum problem

\[
\max_{\mu : |\mu| = \lambda} \int_{B^{2n}(r)} W_{H_\mu} \, dx dy, \tag{2.17}
\]

and denote by \( \tilde{\mu} \) one of its maximizers.

From the sesquilinearity property of the Wigner distribution and Lemma 2.1, we conclude that for every \( R \in O(n) \) one has:

\[
\int_{B^{2n}(r)} W_{H_{\tilde{\mu}}(Rx)} \, dx dy = \sum_\nu |c_\nu(\tilde{\mu}, R)|^2 \int_{B^{2n}(r)} W_{H_\nu} \, dx dy. \tag{2.18}
\]

Since \( H_{\tilde{\mu}} \) is a maximizer, this implies that for any \( \nu_0 \) with \( c_{\nu_0}(\tilde{\mu}, R) \neq 0 \) one has

\[
\int_{B^{2n}(r)} W_{H_{\nu_0}} \, dx dy = \int_{B^{2n}(r)} W_{H_{\tilde{\mu}}(Rx)} \, dx dy = \int_{B^{2n}(r)} W_{H_{\nu_0}} \, dx dy, \tag{2.19}
\]

i.e., \( H_{\nu_0} \) is also a maximizer. The lemma will be proved if we can show that, starting from any \( \tilde{\mu} \), we can, by a succession of rotations and intermediate indices, finally reach any given \( \nu \).
The proof will proceed in two steps. The first is to go from $\tilde{\mu}$, by a succession of two-dimensional rotations, to $(\lambda,0,0,\ldots,0)$ with $\lambda = \sum_{j=1}^{n} \tilde{\mu}_j$.

First, we show that there is a rotation $R' \in O(n)$ with

$$c_{\tilde{\mu}'}(\tilde{\mu}, R') \neq 0,$$

where $\tilde{\mu}' := ((\tilde{\mu}_1 + \tilde{\mu}_2),0,\tilde{\mu}_3,\ldots,\tilde{\mu}_n)$.

Thus, $\tilde{\mu}'$ is also a maximizer. In a similar fashion, we can go from $\tilde{\mu}'$ to $\tilde{\mu}''$, where $\tilde{\mu}'' := ((\tilde{\mu}_1 + \tilde{\mu}_2 + \tilde{\mu}_3),0,0,\tilde{\mu}_4,\ldots,\tilde{\mu}_n)$. Proceeding inductively, we finally arrive at the conclusion that $(\lambda,0,0,\ldots,0)$ is a maximizer.

A rotation $R'$ that accomplishes the first step to $\tilde{\mu}'$ is simply $R': x_1 \rightarrow (x_1 + x_2)/\sqrt{2}$, $x_2 \rightarrow (x_1 - x_2)/\sqrt{2}$, $x_j \rightarrow x_j$ for $j > 2$. The monomial $x_1^{\mu_1} x_2^{\mu_2}$ becomes $\frac{1}{2}(x_1 + x_2)^{\tilde{\mu}_1} (x_1 - x_2)^{\tilde{\mu}_2}$ and this obviously contains the monomial $x_1^{(\tilde{\mu}_1 + \tilde{\mu}_2)}$ with a non-zero coefficient.

The second step is to go in the other direction, from $(\lambda,0,\ldots,0)$ to $(\nu_1,\nu_2,\ldots,\nu_n)$ when $\sum_{j=1}^{n} \nu_j = \lambda$. As before, we do this with a sequence of two-dimensional rotations, the first of which takes us from $(\lambda,0,\ldots,0)$ to $(\lambda - \nu_2, \nu_2,0,\ldots,0)$. From thence we go to $(\lambda - \nu_2 - \nu_3, \nu_2, \nu_3, 0, \ldots, 0)$, and so forth. This can be accomplished with the same rotation as before, namely $R': x_1 \rightarrow (x_1 + x_2)/\sqrt{2}$, $x_2 \rightarrow (x_1 - x_2)/\sqrt{2}$, $x_j \rightarrow x_j$ for $j > 2$.

\[\square\]

**Acknowledgements:** We thank P. Flandrin, A. J. E. M. Janssen, and F. Luef for helpful discussions. This work was partially supported by U.S National Science Foundation grants PHY-0965859 (E. H. L.), and (DMS-0635607) (Y. O.).

**References**


Elliott H. Lieb  
Departments of Mathematics and Physics, Princeton University. P.O. Box 708, Princeton NJ 08542, USA  
Email: lieb@princeton.edu

Yaron Ostrover  
School of Mathematics, Institute for Advanced Study, Princeton NJ 08540, USA  
Email: ostrover@ias.edu