Non-Euclidean Geometry (spring 2011)

Exercise No. 1

1. Let $C(x, r) \subset \mathbb{R}^2$ be the circle with center x and radius r, and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry. Prove that $f(C(x, r)) = C(f(x), r)$.

2. Prove that any two circles in the plane intersect in exactly 0,1,2 points, and that if they intersect in two points then the line connecting their centers is the perpendicular bisector of the line segment connecting the points where the two circles intersect.

3. Let $P = (0, 0), Q = (1, 0), R = (0, 1)$ in $\mathbb{R}^2$. Using only the definition of distance, show that if S and T are two points satisfying
$$\|S - P\| = \|T - P\|, \ \|S - Q\| = \|T - Q\|, \ \|S - R\| = \|T - R\|,$$
then $S = T$. Moreover, let f, g be two isometries of $\mathbb{R}^2$ such that $f(P) = g(P)$, $f(Q) = g(Q)$, and $f(R) = g(R)$. Show that $f \equiv g$. What happen in higher dimensions?

4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the isometry defined by $f(x, y) = (y + 1, x + 1)$. Decide whether f is a translation, rotation, reflection, or a glide reflection.

5. Consider the three lines in the plane given by $L_1 = \{(x, y) \,|\, x + y = 1\}, L_2 = \{(x, y) \,|\, x = 0\}, L_3 = \{(x, y) \,|\, y = 0\}$. Describe the isometry $R_{L_3} R_{L_2} R_{L_1}$ (where R_{L_i} is the reflection with respect to L_i) as either a translation, rotation, reflection, or a glide reflection.

6. Show that every isometry of the plane can be written as a composition of at most three reflections. (Hint: two lemmas that we proved in class might be useful here).

7. Is it possible to represent a glide as a composition of two reflections?

8. Let $\theta \in [0, 2\pi]$. Show that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $M = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{smallmatrix}\right)$ then f is a reflection in a line L through the origin. Find the line of reflection.

9. Let L_1, L_2 two lines intersecting at a point $p \in \mathbb{R}^2$. Let R_{L_i} be the reflection with respect to L_i, $i = 1, 2$. Show that the composition $R_{L_2} \circ R_{L_1}$ is a rotation around P with angle of rotation equal to twice the angle formed by the two intersecting lines.

10. Prove that if $f : \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry such that $f(0) = 0$, then either
$$f(u) \times f(v) = f(u \times v), \text{ or } f(u) \times f(v) = -f(u \times v),$$
for all $v, u \in \mathbb{R}^3$. (here "$\times$" denotes the cross product in $\mathbb{R}^3$).