# Non-Euclidean Geometry (spring 2011)

Partial Solutions to Exercise No. 1 - Euclidean Isometries.

**Note: There is often more than one way to solve the homework problems. The solutions presented here are merely some possibilities.**

1. Let $C(x,r) \subset \mathbb{R}^2$ be the circle with center $x$ and radius $r$, and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry. Prove that $f(C(x,r)) = C(f(x), r)$.

   **Solution:** Note that for an isometry $f : \mathbb{R}^2 \to \mathbb{R}^2$ one has

   $$f(C) = \{f(z) \mid d(x,z) = r\} = \{f(z) \mid d(f(x), f(z)) = r\}$$

   Thus, every point of $f(C)$ is a distance $r$ from $f(x)$. Therefore, $f(C) \subseteq C(f(x), r)$. On the other hand,

   $$\begin{aligned} C(f(x), r) &= \{f(f^{-1}(x)) \mid x \in C(f(x), r)\} = \{f(y) \mid y \in f^{-1}(C(f(x), r))\} \\ &\subseteq \{f(z) \mid z \in C\} = f(C) \end{aligned}$$

2. Prove that any two circles in the plane intersect in exactly 0,1,2 points, and that if they intersect in two points then the line connecting their centers is the perpendicular bisector of the line segment connecting the points where the two circles intersect.

   **Solution I (without coordinates) :** Take $C_1$, $C_2$ distinct circles of radii $r_1$ and $r_2$ around $P_1$ and $P_2$. If $P_1 = P_2$ and $r_1 \neq r_2$ they don't intersect. Assume $P_1 \neq P_2$. Let $L$ be the line containing $P_1$ and $P_2$ and let $f$ be a reflection through $L$. If $P$ is a point of intersection of $C_1$ and $C_2$ then $P' = f(P)$ is also a point of intersection since the circles are mapped onto themselves by $f$. So, if $f(P) = P$ then $P$ lies on $L$ and the two circles are tangent. If $f(P) \neq P$, then $P$ does not lie on $L$. From the triangle inequality we get $d(P_1, P_2) < d(P_1, P) + d(P, P_2) = r_1 + r_2$. Thus, if $d(P_1, P_2) > r_1 + r_2$ then the circles don't intersect. Moreover, if $C_1$ and $C_2$ intersect at two points $P$ and $P'$, then the line $L$ joining $P_1$ and $P_2$ is the perpendicular bisector of $PP'$. Indeed, let $Q$ be the intersection of $L$ and $PP'$. Then $P' = f(P)$, $PP' \perp L$ and $d(Q,P) = d(Q,P')$ from the definition of reflection.

   **Solution II (with coordinates):** Assume for simplicity that $C_1$ is given by $x^2 + y^2 = r_1^2$, and $C_2$ is given by $(x-d)^2 + y^2 = r_2^2$. A simple algebraic manipulation gives $r_1^2 = r_2^2 + 2xd - d^2$. Solving for $x$ gives $x = (r_1^2 - r_2^2 + d^2)/2d$. Substituting into the equation for $C_1$ and solving for $y$ gives $y = \pm\sqrt{r_1^2 - \left(\frac{r_1^2 - r_2^2 + d^2}{2d}\right)^2}$, which has either zero, one or two solutions.

3. Let $P = (0,0), Q = (1,0), R = (0,1)$ in $\mathbb{R}^2$. Using only the definition of distance, show that if $S$ and $T$ are two points satisfying

$$\|S - P\| = \|T - P\|, \ \|S - Q\| = \|T - Q\|, \ \|S - R\| = \|T - R\|,$$

then $S = T$. Moreover, let $f, g$ be two isometries of $\mathbb{R}^2$ such that $f(P) = g(P)$, $f(Q) = g(Q)$, and $f(R) = g(R)$. Show that $f \equiv g$. What happens in higher dimensions?

**Solution:** (a) Let $S = (x, y)$ and $T = (x', y')$. It follows from the given data that: $(i)$ $x^2 + y^2 = (x')^2 + (y')^2$, $(ii)$ $(x-1)^2 + y^2 = (x'-1)^2 + (y')^2$, $(iii)$ $x^2 + (y-1)^2 = (x')^2 + (y'-1)^2$. From the first two equations one has $x = x'$ and from the last two $y = y'$, and hence $S = T$.

(b) Write $f(u) = Au + b$ and $g(u) = Bu + c$, where $AA^T = I = BB^T$ and $b, c \in \mathbb{R}^2$. Then, $f(P) = g(P) \Rightarrow b = c$, $f(Q) = g(Q) \Rightarrow Ae_1 + b = Be_1 + c \Rightarrow Ae_1 = Be_1$, and $f(R) = G(R) \Rightarrow Ae_2 + b = Be_2 + c \Rightarrow Ae_2 = Be_2$. Thus $A = B$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the isometry defined by $f(x, y) = (y + 1, x + 1)$. Decide whether $f$ is a translation, rotation, reflection, or a glide reflection.

**Solution:** The map $g(x, y) = (y, x)$ is a linear transformation with $\det g = -1$, hence a reflection. Since $f(x, y) = g(x, y) + (1, 1)$, is a composition of a reflection and a translation, it is a glide.

5. Consider the following three lines in the plane: $L_1 = \{(x, y) \,|\, x + y = 1\}, L_2 = \{(x, y) \,|\, x = 0\}$, and $L_3 = \{(x, y) \,|\, y = 0\}$. Describe the isometry $R_{L_3} R_{L_2} R_{L_1}$ (where $R_{L_i}$ is the reflection with respect to $L_i$) either as a translation, rotation, reflection, or a glide reflection. (Hint: draw a picture first.)

**Solution:** Rotate $L_2$ and $L_3$ about their common point $(0, 0)$ by $\pi/4$ in the positive sense, so that the new line $L_2'$ is parallel to $L_1$. Then, $L_2' = \{(x, y) \mid x + y = 0\}$ and $L_3' = \{(x, y) \mid x - y = 0\}$. Next, since $R_{L_3} R_{L_2} = R_{L_3'} R_{L_2'}$ (note that the angles between both pairs of lines is the same) one has $R_{L_3} R_{L_2} R_{L_1} = R_{L_3'} R_{L_2'} R_{L_1}$. But $R_{L_2'} R_{L_1}$ is translation by the vector $(-1, -1)$, and $L_3'$ is parallel to this vector. Thus, by definition $R_{L_3'} R_{L_2'} R_{L_1}$ is the glide reflection associated to the line $L_3'$ and the translation $(-1, -1)$.

6. Show that every isometry of the plane can be written as a composition of at most three reflections. (Hint: two lemmas that we proved in class might be useful here).

**Solution:** The solution was given in class.

7. Is it possible to represent a glide reflection as a composition of two reflections?

**Solution:** No, the product of two reflections in lines in $\mathbb{R}^2$ is a rotation if the lines meet and a translation otherwise.

8. Let $\theta \in [0, 2\pi]$. Show that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $f(v) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} (v)$, then $f$ is a reflection in a line $L$ through the origin. Find the line of reflection.

9. Let $L_1, L_2$ two lines intersecting at a point $p \in \mathbb{R}^2$. Let $R_{L_i}$ be the reflection with respect to $L_i$, $i = 1, 2$. Show that the composition $R_{L_2} \circ R_{L_1}$ is a rotation around $P$ with angle of rotation equal to twice the angle formed by the two intersecting lines.

   **Solution:** Let $f_i$ denote the reflection in $L_i$, $i = 1, 2$. We wish to show that $f_2 \circ f_1 = R_{P,2\theta}$, where $\theta$ is the angle between $L_2$ and $L_1$. Note first that $f_2 \circ f_1(P) = P$, that for every point $Q$ one has $d(P, f_2 \circ f_1(Q)) = d(Q, P)$, and that $f_2 \circ f_1(Q)$ lies on a circle of radius $d(P, Q)$ around $Q$. So, it is suffices to show that the angle between the line $S$ through $P$ and $Q$ and the line $S'$ through $P$ and $f_2 \circ f_1(Q)$ is $2\theta$. Let $\phi$ be the angle between $L_1$ and $S$. The angle between $f_1(S)$ and $L_1$ is also $\phi$. If $f_1(S)$ lies between $L_1$ and $L_2$ then the angle between $L_2$ and $f_1(S)$ is $\theta - \phi$ as is the angle between $f_2 \circ f_1(S) = S'$ and $L_2$. Therefore the angel between $S$ and $S'$ is $2\phi + 2(\theta - \phi) = 2\theta$. If $f_1(S)$ lies on the other side of $L_2$, then the angle between $f_1(S)$ and $L_2$ is $\psi'$, where $\phi' = \phi - \theta$. In this case, the angle between $L_2$ and $f_2 \circ f_1(S) = S'$ is also $\phi'$ and the angle between $S$ and $S'$ is $2\phi - 2\phi' = 2\theta$.

10. Prove that if $f : \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry such that $f(0) = 0$, then either

$$f(u) \times f(v) = f(u \times v), \text{ or } f(u) \times f(v) = -f(u \times v),$$

   for all $v, u \in \mathbb{R}^3$. (here " $\times$ " denotes the cross product in $\mathbb{R}^3$).

   **Solution:** We know that $f(u) = Au$ for some matrix $A \in O(3)$. Thus, the set $\{Ae_1, Ae_2, Ae_3\}$ consists of the columns of $A$ and so is an orthonormal basis for $\mathbb{R}^3$. For any orthonormal basis $\{v_1, v_2, v_3\}$, and $w \in \mathbb{R}^3$ we have $w = \sum_{i=1}^3 (w \cdot v_i)v_i$. Taking $w = Au \times Av$, and $v_i = Ae_i$, we conclude:

$$\begin{aligned}
Au \times Av &= \sum_{i=1}^{3}((Au \times Av) \cdot Ae_i)Ae_i = \sum_{i=1}^{3} \det\,[Au\,Av\,Ae_i]Ae_i = \\
&= \sum_{i=1}^{3} \det\,(A[u\,v\,e_i])Ae_i = \sum_{i=1}^{3}(\det A)(\det\,[u\,v\,e_i])Ae_i \\
&= (\det A)\sum_{i=1}^{3}((u \times v) \cdot e_i)Ae_i
\end{aligned}$$

   On the other hand,

$$A(u \times v) = A(\sum_{i=1}^{3}((u \times v) \cdot e_i)e_i) = \sum_{i=1}^{3}((u \times v) \cdot e_i)Ae_i$$

   therefore $Au \times Av = \det(A)A(u \times v)$ and the proof is complete.