Non-Euclidean Geometry (spring 2011)

Exercise No. 10 - Projective Geometry

1. Show that three points in a projective space of dimension at least 2 lie on a projective plane which is unique unless the points are collinear.

2. Let P, P' be distinct projective planes in a 3-dimensional projective space. Show that $P \cap P'$ is a projective line

3. Here is a "cheap proof" of Desargues Theorem in the case when $dim\mathbb{P}(V) \geq 3$ and the triangles ABC and $A'B'C'$ lie in different planes P and P': (i) Prove that P, A, A', B, B', C, C' lie in a 3-dimensional projective subspace (so that we can restrict attention to that subspace and assume $dim\mathbb{P}(V) = 3$). (ii) Show that the intersections Q, R, S all lie in $P \cap P'$ which is a line by the previous question.

4. Let $E_1 = \mathbb{P}(U_1)$ and $E_2 = \mathbb{P}(U_2)$ be two hyperplanes in a projective space $\mathbb{P}(V)$ (for example, two lines in a projective plane) and let $W\ in\mathbb{P}(V)$ be a point not in E_1 or E_2. Then the central projection from E_1 to E_2 with center W is the map $\hat{f} : E_1 \to E_2$ that maps a point $A \in E_1$ to the intersection of E_2 with the line through W and A. Show that the central projection $\hat{f}$ is a projective transformation $E_1 \to E_2$.

5. Let $\mathbb{P}(V)$ and $\mathbb{P}(W)$ be two n-dimensional projective spaces and suppose $A_1, \ldots, A_{n+2} \in \mathbb{P}(V)$ and $B_1, \ldots, B_{n+2} \in \mathbb{P}(W)$ are in general position. Then there exists a unique projective transformation $\hat{f} : \mathbb{P}(V) \to \mathbb{P}(W)$ with $\hat{f}(A_i) = B_i$ for $i = 1, \ldots, n+2$.

6. Let L_1, L_2 be distinct projective lines in a projective plane that intersect at a point A. Let $\tau : L_1 \to L_2$ be a projective transformation such that $\tau A = A$. Show that τ is a projection from some point of the plane. Hint: Choose B, C on L_1 distinct from A and let $B' = \tau B$, $C' = \tau C$. If τ was a projection, where would its centre have to lie?

7. Let L_1 and L_2 be distinct projective lines in a projective plane and $\tau : L_1 \to L_2$ a projective transformation. We are going to show that τ is a composition of two projections. Let A, B, C be distinct points on L_1 and let $A' = \tau A$, $B' = \tau B$, and $C' = \tau C$. Without loss of generality, assume that neither A or A' are in $L_1 \cap L_2$. Set $P = AB' \cap A'B,\ Q = AC' \cap A'C$ and let L' be the line PQ. Let $\tau_1 : L_1 \to L'$ be the projection with centre A' and $\tau_2 : L' \to L_2$ be the projection with centre A. (a) Prove that $\tau = \tau_2 \circ \tau_1$. Hint: What does $\tau_2 \circ \tau_1$ do to A, B, C?

8. (*) Prove Brianchon's Theorem: Let the sides $AB', B'C, CA', A'B, BC', C'A$ of a hexagon pass alternately through two (different) points P and Q in a projective plane. Then the lines joining opposite vertices AA', BB', CC' are concurrent. Hint: choose a basis so that P, C, Q, C' have homogeneous coordinates $[1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1]$.