Non-Euclidean Geometry (spring 2011)

Partial Solutions to Exercise No. 10 - Projective Geometry

1. Show that three points in a projective space of dimension at least 2 lie on a projective plane which is unique unless the points are collinear.

   Solution: Let the three points be $[v0], [v1], [v2]$ and contemplate the span U of the v_i. Then U is at most 3-dimensional. If $dim(U) = 3$ then our points lie in the projective plane $\mathbb{P}(U)$ and if $\mathbb{P}(W)$ is another such projective plane then $U \subset W$ and $dim(U) = dim(W)$ hence $U = W$ so the plane is unique. If $dim(U) \leq 2$ then the three points are collinear since the v_i lie in some 2-dimensional subspace (which we could take to be U if $dim(U) = 2$). Now take any 3-dimensional subspace W containing the v_i (there are many of these) to see that our points lie in a plane $\mathbb{P}(W)$.

2. Let P, P' be distinct projective planes in a 3-dimensional projective space. Show that $P \cap P'$ is a projective line

   Solution: Let $P = \mathbb{P}(U),\ P' = \mathbb{P}(U')$ be planes in the 3-dimensional projective space $\mathbb{P}(V)$. Thus $U, U' \subset V$ are vector spaces with $\dim V = 4$ and $\dim(U) = \dim(U') = 3$. Then

   $$4 \geq dim(U + U') = dimU + dimU' - dim(U \cap U') = 6 - dim(U \cap U'),$$

   yielding $dim(U \cap U') \geq 2$. On the other hand, $dim(U \cap U') \leq dimU = 3$ with equality if and only if $U = U'$ which cannot happen since P, P' are distinct. Thus, $dim(U \cap U') = 2$ so that $P \cap P' = \mathbb{P}(U \cap U')$ is a projective line.

3. Here is a "cheap proof" of Desargues Theorem in the case when $dim\mathbb{P}(V) \geq 3$ and the triangles ABC and $A'B'C'$ lie in different planes P and P': (i) Prove that P, A, A', B, B', C, C' lie in a 3-dimensional projective subspace (so that we can restrict attention to that subspace and assume $dim\mathbb{P}(V) = 3$). (ii) Show that the intersections Q, R, S all lie in $P \cap P'$ which is a line by the previous question.

   Solution: (i) P, A, A', B, B' all lie in the plane $\pi = \mathbb{P}(W)$ containing the lines AA' and BB'. The remaining points lie on the line $CC' = \mathbb{P}(U)$ which intersects this plane exactly at the point P (otherwise CC' lies in the plane which then contains both triangles contrary to assumption). The join of this line and plane is then$\mathbb{P}(U + W)$ which is 3-dimensional since $dim(U + W) = 2 + 3 - 1 = 4$.

   (ii) Consider $Q = BC \cap B'C'$: both B and C lie in the plane P whence the line BC lies in P also. Similarly, $B'C'$ lies in P' so that $Q = BC \cap B'C'$ lies in $P \cap P'$. The same argument shows that R and S also lie on the line $P \cap P'$.

4. Let $E_1 = \mathbb{P}(U_1)$ and $E_2 = \mathbb{P}(U_2)$ be two hyperplanes in a projective space $\mathbb{P}(V)$ (for example, two lines in a projective plane) and let $W in \mathbb{P}(V)$ be a point not in E_1 or E_2. Then the central projection from E_1 to E_2 with center W is the map $\hat{f} : E_1 \to E_2$ that maps a point $A \in E_1$ to the intersection of E_2 with the line through W and A. Show that the central projection $\hat{f}$ is a projective transformation $E_1 \to E_2$.

   Solution: We have to show that $\hat{f}$ comes from an invertible linear map $f : U_1 \to U_2$. Note that W, as point in $\mathbb{P}(V)$, is a 1-dimensional subspace of V . Since it does not lie in E_2 one has, $W \cap U_2 = \{0\}$. This means that V is the direct sum $V = W \oplus U_2$, and there are two linear maps $p_W : V \to W$ and $p_{U_2} : V \to U_2$ (the projections onto W and U_2) such that for any $v \in V$, $p_W(v)$ and $p_{U_2}(v)$ are the unique vectors in W and U_2 such that $v = p_W(v) + p_{U_2}(v)$. Claim: The central projection $\hat{f}$ comes from the linear map $p_{U_2}|U_1$, the restriction of p_{U_2} to U_1. To see this, let $a \in U_1$ be a representative vector of $A \in E_1$. Then $p_{U_2}(a) \neq 0$, because $p_{U_2}(a) = 0$ would mean $a \in W$, but $U_1 \cap W = \{0\}$ because by assumption E_1 does not contain W. This shows that $p_{U_2}|U_1$ is invertible, because it an injective linear map $U_1 \to U_2$ and $dim U_1 = dim U_2$. Now $p_{U_2}(a) \in U_2$,so $[p_{U_2}(a)] \in E_2$. Also $a = p_W(a) + p_{U_2}(a)$,or $p_{U_2}(a) = a - p_W(a)$, so $p_{U_2}(a) \in [a] + W$, which means that$[p_{U_2}(a)]$ is in the (projective) line through $A \in \mathbb{P}(V)$ and $W \in \mathbb{P}(V)$. Hence $[p_{U_2}(a)]$ is the intersection of E_2 with the line through W and A, so it is the image of A under the central projection.

5. Let $\mathbb{P}(V)$ and $\mathbb{P}(W)$ be two n-dimensional projective spaces and suppose $A_1, \ldots, A_{n+2} \in \mathbb{P}(V)$ and $B_1, \ldots, B_{n+2} \in \mathbb{P}(W)$ are in general position. Then there exists a unique projective transformation $\hat{f} : \mathbb{P}(V) \to \mathbb{P}(W)$ with $\hat{f}(A_i) = B_i$ for $i = 1, \ldots, n+2$.

   Solution: Existence: By a lemma we proved in class, we may choose representative vectors $a_1, \ldots, a_{n+2}$ for $A_1, \ldots, A_{n+2}$ and $b_1, \ldots, b_{n+2}$ for $B_1, \ldots, B_{n+2}$ such that $\sum_{i=1}^{n+1} a_i = a_{n+2}$ and $\sum_{i=1}^{n+1} b_i = b_{n+2}$. Also by the general position assumption, $a_1, \ldots, a_{n+1}$ and $b_1, \ldots, b_{n+1}$ are bases of V and W, respectively. Hence there is an invertible linear map $f : V \to W$ with $f(a_i) = b_i$ for $i = 1, \ldots, n+1$. But then also

$$f(a_{n+2}) = f(\sum_{i=1}^{n+1} a_i) = \sum_{i=1}^{n+1} f(a_i) = \sum_{i=1}^{n+1} b_i = b_{n+2}$$

   So f maps the 1-dimensional subspaces $A_i = [a_i] \subset V$ to $B_i = [b_i] \subset W$ for $i = 1, \ldots, n+2$.

   Uniqueness: Let $g : V \to W$ be another invertible linear map with $g(a_i) \in B_i$ for $i = 1, \ldots, n+2$. Then $\widetilde{b_i} = g(a_i)$ would be another set of representative vectors for the B_i with

$$\widetilde{b}_{n+2} = g(a_{n+2}) = g(\sum_{i=1}^{n+1} a_i = \sum_{i=1}^{n+1} g(a_i) = \sum_{i=1}^{n+1} \widetilde{b}_i$$

   By the uniqueness part of the lemma mentioned above, this implies $\widetilde{b_i} = \lambda b_i$ for some $\lambda \neq 0$, so $g = \lambda f$, and g and f induce the same projective transformation $\mathbb{P}(V) \to \mathbb{P}(W)$.

6. Let L_1, L_2 be distinct projective lines in a projective plane that intersect at a point A. Let $\tau : L_1 \to L_2$ be a projective transformation such that $\tau A = A$. Show that τ is a projection from some point of the plane. Hint: Choose B, C on L_1 distinct from A and let $B' = \tau B$, $C' = \tau C$. If τ was a projection, where would its centre have to lie?

   Solution: Let $P = BB' \cap CC'$ and contemplate the projection $\tau_1 : L_1 \to L_2$ with centre P. By construction, we have $\tau_1 1(B) = B'$ and $\tau_1(C) = C'$ but, also, $\tau_1(A) = A$. Thus τ and τ_1 agree on three distinct points of L_1 (thus three points in general position) and so agree everywhere: $\tau = \tau_1$ so that τ is a projection.

7. Let L_1 and L_2 be distinct projective lines in a projective plane and $\tau : L_1 \to L_2$ a projective transformation. We are going to show that τ is a composition of two projections. Let A, B, C be distinct points on L_1 and let $A' = \tau A$, $B' = \tau B$, and $C' = \tau C$. Without loss of generality, assume that neither A or A' are in $L_1 \cap L_2$. Set $P = AB' \cap A'B$, $Q = AC' \cap A'C$ and let L' be the line PQ. Let $\tau_1 : L_1 \to L'$ be the projection with centre A' and $\tau_2 : L' \to L_2$ be the projection with centre A. (a) Prove that $\tau = \tau_2 \circ \tau_1$. Hint: What does $\tau_2 \circ \tau_1$ do to A, B, C?

   Solution: Let $R = AA' \cap L'$. Then $\tau_1(A) = R$, $\tau_1(B) = P$ and $\tau_1(C) = Q$. Moreover, $\tau_2(R) = A'$, $\tau_2(P) = B'$ and $\tau_2(Q) = C'$. Thus τ and $\tau_2 \circ \tau_1$ agree on the three points A, B, C in general position and so everywhere.

8. (*) Prove Brianchon's Theorem: Let the sides $AB', B'C, CA', A'B, BC', C'A$ of a hexagon pass alternately through two (different) points P and Q in a projective plane. Then the lines joining opposite vertices AA', BB', CC' are concurrent. Hint: choose a basis so that P, C, Q, C' have homogeneous coordinates $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$, $[1, 1, 1]$.