Non-Euclidean Geometry (spring 2011)

Partial Solutions to Exercise No. 2 - Subgroups of Isometries and More

1. If G is a discrete subgroup of Iso($\mathbb{R}$), prove that G contains a translation T by a minimum distance, and that every translation in G is a power of T.

   Solution: If a subgroup of $I(\mathbb{R})$ contained translations by arbitrarily small amounts it would not be discrete. Hence there is a translation T_m by some smallest number m. Then if there is a translation T_k by an amount which was not a multiple of m, we could write $k = am + r$ with $|r| < |m|$ and T_r would also be in the subgroup, giving a contradiction.

2. Prove that the group $SO(3)$ is the group of all rotations about straight lines through 0 in $\mathbb{R}^3$. (hint: start with the fact that the determinant of a matrix is the product of its eigenvalues, and deduce that at least one of them is $+1$.)

   Solution: First note that for any matrix, the determinant is the product of the eigenvalues. Next, any 3×3 matrix has a real eigenvalue λ since the polynomial you have to solve to find the eigenvalues is a cubic and any cubic has a real root. Since the transformation is length preserving we must have $\lambda = \pm 1$. If the other eigenvalues are real too, they must also be ± 1 and since their product is ± 1, one of the eigenvalues is $+1$. If the other eigenvalues are non-real, they are complex conjugate to one another and so their product is positive. So the real eigenvalue is $+1$. Thus in either case, the transformation T has a fixed vector a with $T(a) = +1a$. (This is called an axis.) In the plane perpendicular to a, vectors in this plane are mapped into this plane (since angles are preserved) and so the transformation acts like an element of $SO(2)$ and it is therefore rotation by some angle about the axis a.

3. Prove that an element of the orthogonal group $O(3)$ is either: a rotation about some axis a, a reflection in some plane, or a rotation about an axis a followed by a reflection in a plane perpendicular to a.

   Solution: The proof is similar to the proof that $SO(3)$ is the group of rotations in $\mathbb{R}^3$. An element T of $O(3)$ has an eigenvalue which is either ± 1. If $\lambda = +1$ then T has a fixed vector a and acts like an element of $O(2)$ in the plane perpendicular to a. If it acts as a rotation here, then we have a rotation in $\mathbb{R}^3$ (case (a)). If it acts as a reflection in a vector b here, then we have a reflection in the plane containing a and b (case (b)). If T has no positive eigenvalue then as above we have $\lambda = -1$ with eigenvector c . As before T acts as an element of $O(2)$ in the plane perpendicular to c and it cannot act as a reflection here otherwise it would have a fixed vector (which would be an eigenvector of $+1$). Hence it acts as a rotation in this plane, and we have case (c).

4. Write explicitly the group tables of C_3 and D_3.

   Solution: See e.g., http://en.wikipedia.org/wiki/Dihedral_group

5. Let $G \subset \mathrm{Iso}(\mathbb{R}^2)$ be a finite subgroup. Show that G has a fixed point: there exists $x \in \mathbb{R}^2$ such that $gx = x$ for all $g \in G$. Conclude that such a G is isomorphic to a finite subgroup of $O(2)$

   Solution: In class we showed that every isometry f of $\mathbb{R}^n$ is of the form $x \to Ax + v$, where $A \in O(n)$ and $v \in \mathbb{R}^n$. This means that f is affine i.e.,

   $$f(\sum \lambda_i x_i) = \sum \lambda_i f(x_i), \quad \text{provided that } \sum \lambda_i = 1$$

   Next, given a finite group $G \subset \mathrm{Iso}(\mathbb{R}^2)$, choose any point $x \in \mathbb{R}^n$, and define the centroid of the orbit of x to be:

   $$c_x = \frac{1}{|G|} \sum_{T \in G} T(x)$$

   Finally, use the fact that f is affine, and that G is a group to show that all of the elements of G fix the point c_x. (Note also that if G contains a translation or a glide reflection then it can't be finite).

6. Let $G \subset \mathrm{Iso}(\mathbb{R}^2)$ be a subgroup. A stabilizer H_x of a point $x \in \mathbb{R}^2$ is defined as follows: $H_x = \{g \in G \,|\, gx = x\}$. Show that H_x is a subgroup of G. Show that if x and y belong to the same orbit of G, the groups H_x and H_y are isomorphic. Assume that G is discrete. Prove that each H_x is isomorphic to a finite subgroup of $O(2)$.

7. Use the previous questions to complete the proof of the following theorem (by L. da Vinci): The groups C_n and D_n, for some $n \in \mathbb{N}$, are the only finite subgroups of $\mathrm{Iso}(\mathbb{R}^2)$.

   Solution: The proof of the theorem above was given in class (using the fact that a finite group of isometries of R^2 must have a fixed point).

8. Find a condition for the product of two rotations in $\mathrm{Iso}(\mathbb{R}^2)$ to be a translation.

   Solution: If $f_1 = \mathrm{Rot}_{a,\alpha}$ and $f_2 = \mathrm{Rot}_{b,\beta}$ then $f_1 \circ f_2$ is a translation if and only if $\alpha = -\beta$. To prove that, write $f_1 = T_a \circ L \circ T_{-a}$ and $f_2 = T_b \circ L' \circ T_{-b}$ and note that if $\alpha a = -\beta$ then $L' = L^{-1}$. Then $f_1 \circ f_2 = a + L(-a + b + L^{-1}(-b + x)) = a + L(b - a) - b + x$ which is translation by the vector $a + L(b - a) - b$. In fact, the product of two rotations centered on a and b with angles α and β is equal to a rotation centered on c, where c is the intersection of: (i) the line ab rotated around a by $-\alpha/2$ and (ii) the line ab rotated around b by $\beta/2$. (so if $\alpha + \beta = 0$, then its a translation). This could also be proven easily with analytic geometry.

9. Prove that the composite of a rotation about a point a and reflection R_L in a line L is a reflection if and only if a lies on the line L and a glide reflection otherwise.

   Solution: The product of a rotation and reflection is either a reflection or a glide reflection. If $a \in L$ then there is a fixed point and we have a reflection. If $a \notin L$, then write the rotation about a as the product of reflection in a line M parallel to L and reflection in a line N. So $f = R_L \circ \mathrm{Rot}_a = R_L \circ R_M \circ R_N = T_b \circ R_N$ where b is a vector perpendicular to L. Thus since b is not perpendicular to N, this is a glide reflection.

10. If $ABCD$ is a rectangle, show that $G_{DA} \circ G_{CD} \circ G_{BC} \circ G_{AB} = \mathrm{Id}$, where G_{AB} is the he glide reflection along a (directed) line AB, and Id is the identity. Find a condition that ensures that gliding around a general quadrilateral in the plane is the identity.

    Solution: For the first part of the question, draw a picture, take a pair of axis a_1 and a_2 at the point A (say a_1 points "north" and a_2 "east") and look at their images under the glides. For the second part of the question: The composite of a pair of glides is a rotation (about some point) by twice the angle between them. So $G_{BC} \circ G_{AB} =$ rotation by 2β and $G_{DA} \circ G_{CD} =$ rotation by 2δ. So if $2\beta + 2\delta \neq 2\pi$, the composite cannot be the identity. If $\beta + \delta = \pi$ then, since A is fixed by the composite, we do have the identity. This is the condition that the quadrilateral is cyclic (the four vertices lie on a circle).

11. Prove that every isometry in $\mathrm{Iso}(\mathbb{R}^3)$ is either: a rotation (about any line in $\mathbb{R}^3$), a translation, a screw translation (a rotation about some line followed by a translation parallel to that line), a reflection (about any plane in $\mathbb{R}^3$), a glide reflection (reflection in a plane Π followed by a translation parallel to Π), a rotatory reflection (a reflection in a plane Π followed by a rotation about an axis perpendicular to Π).