Non-Euclidean Geometry (spring 2011)

Exercise No. 2 - Subgroups of Isometries and More

1. If G is a discrete subgroup of $\mathrm{Iso}(\mathbb{R})$, prove that G contains a translation T by a minimum distance, and that every translation in G is a power of T.
2. Prove that the group $SO(3)$ is the group of all rotations about straight lines through 0 in $\mathbb{R}^3$. (hint: start with the fact that the determinant of a matrix is the product of its eigenvalues, and deduce that at least one of them is $+1$.)
3. Prove that an element of the orthogonal group $O(3)$ is either: a rotation about some axis a, a reflection in some plane, or a rotation about an axis a followed by a reflection in a plane perpendicular to a.
4. Write explicitly the group tables of C_3 and D_3.
5. Let $G \subset \mathrm{Iso}(\mathbb{R}^2)$ be a finite subgroup. Show that G has a fixed point: there exists $x \in \mathbb{R}^2$ such that $gx = x$ for all $g \in G$. Conclude that such a G is isomorphic to a finite subgroup of $O(2)$
6. Let $G \subset \mathrm{Iso}(\mathbb{R}^2)$ be a subgroup. A stabilizer H_x of a point $x \in \mathbb{R}^2$ is defined as follows: $H_x = \{g \in G \,|\, gx = x\}$. Show that H_x is a subgroup of G. Show that if x and y belong to the same orbit of G, the groups H_x and H_y are isomorphic. Assume that G is discrete. Prove that each H_x is isomorphic to a finite subgroup of $O(2)$.
7. Use the previous questions to complete the proof of the following theorem (by L. da Vinci): The groups C_n and D_n, for some $n \in \mathbb{N}$, are the only finite subgroups of $\mathrm{Iso}(\mathbb{R}^2)$.
8. Find a condition for the product of two rotations in $\mathrm{Iso}(\mathbb{R}^2)$ to be a translation.
9. Prove that the composite of a rotation about a point a and reflection R_L in a line L is a reflection if and only if a lies on the line L and a glide reflection otherwise.
10. If $ABCD$ is a rectangle, show that $G_{DA} \circ G_{CD} \circ G_{BC} \circ G_{AB} = \mathrm{Id}$, where G_{AB} is the he glide reflection along a (directed) line AB, and Id is the identity. Find a condition that ensures that gliding around a general quadrilateral in the plane is the identity.
11. Prove that every isometry in $\mathrm{Iso}(\mathbb{R}^3)$ is either: a rotation (about any line in $\mathbb{R}^3$), a translation, a screw translation (a rotation about some line followed by a translation parallel to that line), a reflection (about any plane in $\mathbb{R}^3$), a glide reflection (reflection in a plane Π followed by a translation parallel to Π), a rotatory reflection (a reflection in a plane Π followed by a rotation about an axis perpendicular to Π).