Non-Euclidean Geometry (spring 2011)

Exercise No. 3 - Isometries, Symmetry Groups, and Spherical Geometry

1. Let $G < \text{Iso}(\mathbb{R}^2)$ be a subgroup, and let $S_x = \{g \in G \mid gx = x\}$ be the stabilizer of $x \in \mathbb{R}^2$. Show that S_x is a subgroup of G, that S_x is isomorphic to S_y if x and y belong to the same orbit of G, and that if G is discrete, then S_x is isomorphic to a finite subgroup of $O(2)$.

2. Let $G < \text{Iso}(\mathbb{R}^3)$ be a finite subgroup. Recall that in the lecture we proved that there are either 2 or 3 orbits of the action of G on the (finite) set P_G of poles on S^2.

   a) Show that when there are only two orbits, G is the cyclic group C_n generated by rotation by $2\pi/n$. Hint, use the formula we proved in class to compute the size of the corresponding stabilizers.

   b) Complete the details of the proof we gave in the lecture, and show that when there are 3 different orbits, one has

S_1	$\|\Omega_1\|$	S_2	$\|\Omega_2\|$	S_3	$\|\Omega_3\|$	$\|G\|$
N	2	2	N	2	N	$2N$
3	4	3	4	2	6	12
4	6	3	8	2	12	24
5	12	3	20	2	30	60

   c) Show that in the first case above, $G = D_n$, and that in the last three cases, G correspond to the symmetry group of the tetrahedron, octahedron (or the cube) and the icosahedron (or the dodecahedron) respectively.

3. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $f(v) = Av$ where

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

   Show that f is an isometry of $\mathbb{R}^3$, and that $f(S^2) = S^2$. Describe the isometry type of f, considered as an isometry of the sphere, in detail. That is, describe it as a reflection in a great circle C, rotation through an angle θ about a point a, or rotation through an angle θ about a point a followed by reflection in the great circle normal to a. Give C, θ and/or a explicitly.

4. (**) For which of the five Platonic bodies can a (countable) collection of copies of the body fill $\mathbb{R}^3$ (without overlaps and the tiling is assumed to be face to face)?

In all the problems below a, b, c are the sides, and α, β, γ are the opposite angles of a spherical triangle, where the radius of the sphere is $R = 1$.

5. Deduce from the spherical cosine theorem we proved in class that $a + b + c < 2\pi$.

6. Prove the following cosine theorem (second cosine theorem) on the sphere

$$\cos\alpha = \cos\beta\,\cos\gamma + \sin\beta\,\sin\gamma\,\cos a$$

7. (*) Prove the spherical sine theorem:

$$\frac{\sin a}{\sin\alpha} = \frac{\sin b}{\sin\beta} = \frac{\sin c}{\sin\gamma}$$

Hint: it might be useful to show first the following lemma (sometimes known as the "three perpendiculars lemma"): Let $A \in \mathbb{R}^3$ be a point outside a plane P, let H be its perpendicular projection on P, and let L be its perpendicular projection on a line l contained in P. Then HL is perpendicular to l.

8. Prove that the medians of a spherical triangle interest at one point.

9. To "solve" a spherical triangle means to find all of the arc, angles, and vertex angles given some of them. Solve a triangle with all three arc angles being $\pi/3$. Solve a triangle with $\angle A = \angle B = \pi/2$ and $\angle C = \pi/3$

10. Let ABC be a spherical triangle and let $A'B'C'$ be its polar triangle. Show that

$$a + \alpha' = b + \beta' = c + \gamma' = a' + \alpha = b' + \beta = c' + \gamma = \pi$$

11. Prove that the polar spherical triangle of $A'B'C'$ is ABC.