Non-Euclidean Geometry (spring 2011)

Partial Solutions to Exercise No. 3

1. Let $G < \text{Iso}(\mathbb{R}^2)$ be a subgroup, and let $S_x = \{g \in G \mid gx = x\}$ be the stabilizer of $x \in \mathbb{R}^2$. Show that S_x is a subgroup of G, that S_x is isomorphic to S_y if x and y belong to the same orbit of G, and that if G is discrete, then S_x is isomorphic to a finite subgroup of $O(2)$.

2. Let $G < \text{Iso}^+(\mathbb{R}^3)$ be a finite subgroup. Recall that in the lecture we proved that there are either 2 or 3 orbits of the action of G on the (finite) set P_G of poles on S^2.

   a) Show that when there are only two orbits, G is the cyclic group C_n generated by rotation by $2\pi/n$. Hint, use the formula we proved in class to compute the size of the corresponding stabilizers.

   b) Complete the details of the proof we gave in the lecture, and show that when there are 3 different orbits, one has

S_1	$\|\Omega_1\|$	S_2	$\|\Omega_2\|$	S_3	$\|\Omega_3\|$	$\|G\|$
N	2	2	N	2	N	$2N$
3	4	3	4	2	6	12
4	6	3	8	2	12	24
5	12	3	20	2	30	60

   c) Show that in the first case above, $G = D_n$, and that in the last three cases, G correspond to the symmetry group of the tetrahedron, octahedron (or the cube) and the icosahedron (or the dodecahedron) respectively.

Solution: First translate so that G fixes the origin. Then each non-identity element of G is a rotation about an axis through 0. Let Ω be the set of unit vectors that are fixed by by some non-identity element of G. Then Ω is a finite set and G acts on it. Let Ω_1,Ω_2,...Ω_J be the different orbits in Ω. The orbitstabilizer theorem shows that each vector $u \in \Omega_j$ has a stabilizer of order $S_j = |G|/|\Omega_j|$. Now we count the number of pairs in the set

$$X = \{(A, u) \ : \ A \in G \setminus I, u \in S^2, \text{ and } Au = u\}$$

Each $A \in G \setminus I$ is a rotation and so fixes exactly two unit vectors. Therefore $|X| = 2(|G| - 1)$. Alternatively, each $u \in \Omega$ gives rise to $|Stab(u)| - 1$ pairs in X. So

$$|X| = \sum_{j=1}^{J} (S_j - 1)|\Omega_j| = \sum_{j=1}^{J} |G| - |\Omega_j|$$

Dividing by $|G|$ we see that

$$2 - \frac{2}{|G|} = \sum_{j=1}^{J} 1 - \frac{1}{S_j} \quad (*)$$

Each stabilizer of $u \in \Omega$ has order at least 2, so $1 - \frac{1}{S_j} \geq \frac{1}{2}$. Hence,

$$2 > 2 - \frac{2}{|G|} = \sum_{j=1}^{J} 1 - \frac{1}{S_j} \geq \frac{1}{2} J,$$

and so J is $1, 2$, or 3. Order the orbits so that $S_1 \geq S_2 \geq S_3 \geq 2$. Clearly there are no solutions to equation $(*)$ above with $J = 1$. If $J = 2$ then $(*)$ gives

$$2 - \frac{2}{|G|} = 2 - \frac{1}{S_1} - \frac{1}{S_2} \leq 2 - \frac{2}{S_1},$$

so $S_1 \geq |G|$. This implies that $S_1 = |G|$ and $|\Omega_1| = 1$. Hence,

$$2 - \frac{2}{|G|} = 2 - \frac{1}{S_1} - \frac{1}{S_2} = 2 - \frac{1}{|G|} - \frac{1}{S_2},$$

and so $S_2 = |G|$ and $|\Omega_2| = 1$. Hence Ω consists of two unit vectors v and $-v$ which are fixed by each isometry in G. The group G is then a finite isometry group of the plane orthogonal to v so it is cyclic by what we proved in class (i.e., the classification of finite subgroup of isometries of $\mathbb{R}^2$. When $J = 3$, equation $(*)$ gives

$$\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} = 1 + \frac{2}{|G|}$$

This implies that

$$\frac{3}{S_3} \geq 1 + \frac{2}{|G|} > 1,$$

so $S_3 = 2$. Equation $(*)$ now yields

$$\frac{2}{S_2} \geq \frac{1}{S_1} + \frac{1}{S_2} = \frac{1}{2} + \frac{2}{|G|} \geq \frac{1}{2}$$

which implies that $S_2 = 2$ or 3. When $S_2 = 2$ we have $\frac{1}{S_1} = \frac{2}{|G|}$ which gives $S_1 = N, S_2 = 2, S_3 = 2$, and $|G| = 2N$. The orbit Ω_1 has has just two points. Let v be one of them. The stabilizer $Stab(v)$ is a finite group of N rotations about v, so it is cyclic generated by a rotation R. Choose u as one of the points in Ω_3. Then the others are $Ru, R^2u, \ldots, R^{N-1}u$. The stabilizer of u has order 2, so it contains a rotation S of order 2. This maps v to $-v$. From this it is not hard to conclude that in this case G is isomorphic to the dihedral group D_n (Note that we are used to thinking of D_n as a group acting on $\mathbb{R}^2$ and containing some reflections. As a subgroup of $Iso(\mathbb{R}^3)$ the elements of order two are not reflections but rotations by π). Next, when $S_2 = 3$ we have

$$\frac{1}{S_1} = \frac{2}{|G|} + \frac{1}{6} > \frac{1}{6},$$

so $S_1 = 3, 4$ or 5. To conclude, we get all the different possibilities that appear in the table of part (b) of the question. Finally, we need to show that these correspond to the symmetry groups of the tetrahedron, octahedron and icosahedron respectively. We will consider the middle case as an example. Here Ω_1 has 6 points. The stabiliser of each is a cyclic group of order $S_1 = 4$. Choose one point $v \in \Omega_1$. The stabiliser of v is a cyclic group of order 4; let R be a generator. Now $-v$ is also fixed by R and has the same stabiliser. So it must be in Ω_1. There remain 4 other points in Ω_1 and these must be w, Rw, R^2w, R^3w all lying in the plane through 0 orthogonal to v. Hence the points of Ω_1 are the 6 vertices of a regular octahedron. Note that the points of Ω_2 are the midpoints of the faces of this octahedron and the points of Ω_3 are the midpoints of the edges. The points of Ω_2 are the vertices of a cube. This is the dual of the octahedron. The polyhedron and its dual have the same symmetry group.

3. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $f(v) = Av$ where

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

   Show that f is an isometry of $\mathbb{R}^3$, and that $f(S^2) = S^2$. Describe the isometry type of f, considered as an isometry of the sphere, in detail. That is, describe it as a reflection in a great circle C, rotation through an angle θ about a point a, or rotation through an angle θ about a point a followed by reflection in the great circle normal to a. Give C, θ and/or a explicitly.

   Solution: It is straigforward to check that $AA^T = I_3$, and hence f is an isometry of $\mathbb{R}^3$. Moureover, since $f(0) = 0$, and f is an isometry of $\mathbb{R}^3$, it preserves dot products (and hence lengths of vectors). Thus f maps S^2 to S^2. Moreover, being an isometry, f has an inverse, so $f(S^2) = S^2$. Now we turn to describe the isometry type of f: Since $\det(A) = -1$, f is orientation reversing. Moreover, it is not hard to check that the only fixed point of f is the origin, which does not lie on the sphere. Thus, f does not fix any point on S^2, and hence must be a rotation about a point a, followed by reflection in the great circle normal to a. To find a, we look for a point such that $f(a) = -a$. This amounts to row-deducing of $[A + I|0]$, for which the general solution is $\text{Span}\{(1, 1, 1)\}$. Since we want a point that lies on the sphere, we take $a = \frac{1}{\sqrt{3}}(1, 1, 1)$ (we could take the minus of this point as well). To find the angle θ of rotation, we take a point in the great circle normal to a, say $v = \frac{1}{\sqrt{2}}(1, -1, 0)$, and compute:

$$\cos\theta = v \cdot f(v) = \frac{1}{2}(1, -1, 0) \cdot (1, 0, -1) = \frac{1}{2} \Rightarrow \theta = \pm\pi/3$$

   To see which angle is correct we compute $v \times f(v) = (1, 1, 1)$, which is in the same direction as a. Thus, the rotation is in the poisitve sense (using the right hand rule, thumb pointing in the direction of a, fingers curl from v to $f(v)$), and so $\theta = \pi/3$.

4. (**) For which of the five Platonic bodies can a (countable) collection of copies of the body fill $\mathbb{R}^3$ (without overlaps and the tiling is assumed to be face to face)?

In all the problems below a, b, c are the sides, and α, β, γ are the opposite angles of a spherical triangle, where the radius of the sphere is $R = 1$.

5. Deduce from the spherical cosine theorem we proved in class that $a + b + c < 2\pi$.

   Solution: It follows from the spherical cosine theorem that we we proved in class that

   $$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha > \cos b \cos c - \sin b \sin c = \cos(b + c)$$

   Looking at the graph of the function $\cos(x)$, for $x \in [0, 2\pi]$, we immediatly conclude that the above inequality implies that $a < b + c < 2\pi - a$.

6. Prove the following cosine theorem (second cosine theorem) on the sphere

   $$\cos \alpha = -\cos \beta \, \cos \gamma + \sin \beta \, \sin \gamma \, \cos a$$

   Solution: When we apply the cosine rule we proved in class to the dual triangle $\triangle^*$ we obtain

   $$\cos a^* = \cos b^* \cos c^* + \sin b^* \sin c^* \cos \alpha^*$$

   Now we can use the result in question 10 below about the relations between angles and side-lengths of $\triangle$ and $\triangle^*$ to conclude that

   $$-\cos \alpha = (-\cos \beta)(-\cos \gamma) + \sin \beta \sin \gamma (-\cos a)$$

7. (*) Prove the spherical sine theorem:

   $$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}$$

   Hint: it might be useful to show first the following lemma (sometimes known as the "three perpendiculars lemma"): Let $A \in \mathbb{R}^3$ be a point outside a plane P, let H be its perpendicular projection on P, and let L be its perpendicular projection on a line l contained in P. Then HL is perpendicular to l.

   Solution: First, I will show a solution that does not use the above hint - I am showing it since it gives an alternative proof of the spherical cosine theorem as well. Let C^* be the reflection of C in the plane spanned by A and B (i.e. reflection in the spherical line through A and B). Note that we may assume that $A = (0, 0, 1)$, $B = (\sin c, 0, \cos c)$ and $C = (\sin b \cos \alpha, \sin b \sin \alpha, \cos b)$. Indeed, we may apply an isometry to move A to the North pole $(0, 0, 1)$. Then a rotation about the z-axis will move the unit tangent vector u to AB at A to $(1, 0, 0)$. Then the point B is distance c from A in this direction, so $B = (\sin c, 0, \cos c)$. The unit tangent vector v to AC at A must be at an angle α to u, so it is $(\cos \alpha, \pm \sin \alpha, 0)$. By reflecting in the xz-plane, if necessary, we may ensure that $v = (\cos \alpha, \sin \alpha, 0)$. The point C is at distance b from A in this direction, so $C = (\cos b)A + (\sin b)v = (\sin b \cos \alpha, \sin b \sin \alpha, \cos b)$ as requested.

Note: from this argument one can conclude the spherical cosine theorem we proved in class by calculating the distance a from B to C directly, i.e.,

$$\cos a = B\cdot C = (\sin c, 0, \cos c)\cdot(\sin b\cos\alpha, \sin b\sin\alpha, \cos b) = \sin b\sin c\cos\alpha + \cos b\cos c$$

Back to the spherical sine theorem: The point C^* is now given by

$$C^* = (\sin b\cos\alpha, -\sin b\sin\alpha, \cos b)$$

The Euclidean distance $\|C - C^*\|$ between C and C^* is equal to $2\sin\frac{1}{2}d(C, C^*)$. So $2\sin\frac{1}{2}d(C, C^*) = 2\sin b\sin\alpha$. If we interchange A and B the point C is unchanged, so we must have $2\sin b\sin\alpha = 2\sin a\sin\beta$. Hence $\sin a/\sin\alpha = \sin b/\sin\beta$.

Here is a more geometric solution: First we prove the "theorem of the three perpendiculars".

Lemma: Let $A \in \mathbb{R}^3$ be a point outside a plane P, let H be its perpendicular projection on P and let L be its perpendicular projection on a line l contained in P. Then HL is perpendicular to l.

Proof of the Lemma: The line l is perpendicular to the plane AHL because it is perpendicular to two nonparallel lines of AHL, namely to AL and AH (to the latter since AH is orthogonal to any line in P). Therefore l is perpendicular to any line of the plane AHL, and in particular to LH.

Now we will use the above to prove the spherical sine theorem: Let H be the projection of A on the plane ABC, let L and M be the projections of A on the lines OB and OC. Then by the lemma, L and M coincide with the projections of H on OB and OC. Therefore,

$$AH = LA\sin\beta = \sin c\sin\beta, \quad AH = MA\sin\gamma = \sin b\sin\gamma$$

Thus, $\sin b/\sin\beta = \sin c/\sin\gamma$. Similarily, by projecting C on the plane AOB and arguing as above, we obtain $\sin b/\sin\beta = \sin a/\sin\alpha$. QED.

8. Prove that the medians of a spherical triangle interest at one point.

   In Euclidean geometry, the fact that the three medians meet at a single point - the centroid of the triangle - is a classical result which goes back to antiquity (see e.g., http://www.cut-the-knot.org/triangle/medians.shtml) for several proofs of this remarkable fact. In the sperical case, let $\triangle = \triangle(ABC)$ be a sperical triangle with 3 (spherical) medians M_A, M_B, M_C connecting A with the mid-point of BC, B with the midpoint of AC, and C with the mid-point of A_B. Next, we look at the projection of these spherical medians to the **plane** triangle ABC (here the projection is from the center of the sphere). It is not hard to check that the image of the medians M_A, M_B and M_C are now medians of the plane triangel ABC and hence the statement follows from a reduction to the Euclidean case.

9. To "solve" a spherical triangle means to find all of the arc, angles, and vertex angles given some of them. Solve a triangle with all three arc angles being $\pi/3$. Solve a triangle with $\angle A = \angle B = \pi/2$ and $\angle C = \pi/3$

Solution: For the first triangle, the spherical cosine theorem gives that $\cos\alpha = 1/3$, which gives $\alpha = \arccos(1/3)$, and by symmetry the other angles are the same. For the second triangle, it's better to use the second cosine law that we proved aboved.

10. Let ABC be a spherical triangle and let $A'B'C'$ be its polar triangle. Show that

$$a + \alpha' = b + \beta' = c + \gamma' = a' + \alpha = b' + \beta = c' + \gamma = \pi$$

Solution: We may apply an isometry to $\triangle$ so that $A = (1, 0, 0)$ and the unit tangent vectors to AB and AC at A are $u = (1, 0, 0)$ and $v = (\cos\alpha, \sin\alpha, 0)$ respectively. Then C^* is perpendicular to the plane through A and B, so it is perpendicular to $A = (0, 0, 1)$ and $u = (1, 0, 0)$. Hence, $C^* = \pm(0, 1, 0)$. Since we also want $C^*B > 0$ we must have $C^* = (0, 1, 0)$. Similarly, $B^* = (\sin\alpha, -\cos\alpha, 0)$. Consequently, $B^* \cdot C^* = -\cos\alpha$ and so the distance from B^* to C^* is $\pi - \alpha$. By interchanging the roles of A, B and C in the above argument we see that the side lengths $a^*,\ b^*,\ c^*$ in $\triangle^*$ are

$$a^* = \pi - \alpha, \quad b^* = \pi - \beta, \quad c^* = \pi - \gamma$$

Now we can use fact (Exercise 11 below) that the triangle $\triangle$ is dual triangle to $\triangle^*$, and so the angles $\alpha^*,\ \beta^*,\ \gamma^*$ must satisfy

$$a = \pi - \alpha^*, \quad b = \pi - \beta^*, \quad c = \pi - \gamma^*$$

This completes the proof.

11. Prove that the polar spherical triangle of $A^*B^*C^*$ is ABC.

Solution: Recall that the dual triangle $\triangle^*$ of $\triangle$ is the spherical triangle with vertices $A^*,\ B^*, C^*$ satisfying:

$$A^* \cdot A > 0; \quad A^* \cdot B = 0; \quad A^* \cdot C = 0;$$

$$B^* \cdot A = 0; \quad B^* \cdot B > 0; \quad B^* \cdot C = 0;$$

$$C^* \cdot A = 0; \quad C^* \cdot B = 0; \quad C^* \cdot C > 0.$$

Note that the unit vector A satisfies $A \cdot B^* = A \cdot C^* = 0$ and $A \cdot A^* > 0$, so the original triangle $\triangle$ is the dual triangle of $\triangle^*$. The vertices of $\triangle^*$ represent the sides of $\triangle$ and the sides of $\triangle^*$ represent the vertices of $\triangle$.