Non-Euclidean Geometry (spring 2011)

Exercise No. 4 - Spherical Geometry and Möbius Transformation

1. Prove that if $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a Euclidean isometry with $f(0) = 0$, and C is a great circle on S^2, then $f(C)$ is a great circle.

2. Show that every isometry of S^2 is either the identity, reflection, rotation, or a rotation around a pair of points; followed by reflection in the great circle orthogonal to them.

3. Describe the isometry $g : S^2 \to S^2$ given by $g(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}(x)$. Hint: determine first how many fixed points g has.

4. Let $v \in S^2$, a great circle C orthogonal to v, and an angle θ. Set f to be the rotation by θ around v, and g a reflection in C. Show that $fg = gf$.

5. (*) Find all spherical triangles with angles $(\alpha, \beta, \gamma) = (\pi/p, \pi/q, \pi/r)$ where p, q, r are positive natural numbers. In each case deduce the number of triangles necessary to tile the sphere and calculate $V - F + E$ for the resulting tessalation.

6. Let $a, b, c, d \in \mathbb{C}$ with $ac - bd \neq 0$, and define a map $T : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ by $T : z \to \frac{az+b}{cz+d}$. Show that these maps (called Möbius transformations) form a group (denoted below by Möb) under composition.

7. Show that the map $\phi : SL(2, \mathbb{C}) \to$ Möb, given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \frac{az+b}{cz+d}$ is a group homomorphism and find its kernel. The group $SL(2, \mathbb{C})/\mathrm{Ker}(\phi)$ (which is isomorphic to Möb) is called the projective special linear group.

8. Show that the Möbius transformation $z \mapsto z + 1$ is conjugate to its inverse in Möb .

9. Let $a, b, c, d \in \mathbb{R}$, let $z \in \mathbb{C}$, and define $f(z) = \frac{az+b}{cz+d}$. Show that if $ad - bc \leq 0$ then $f(z)$ is not a one-to-one (or onto) map from the upper half plane $\mathbb{H}^2$ to itself, and that if $ad - bc > 0$ then $f(z)$ is a one-to-one and onto map from $\mathbb{H}^2 \to \mathbb{H}^2$.

10. (a) Show that any Möbius transformation, other than the identity map, has either one or two fixed points. (b) Let $A \in GL(2, \mathbb{C})$, and let $(z_1, z_2)^T$ be an eigenvector for the matrix A. Show that z_1/z_2 is a fixed point for the Möbius transformation $T_A(z) = \frac{az+b}{cz+d}$. (c) Find the fixed points of $h_b(z) = z + b, k_a(z) = az$, and $i(z) = 1/z$, where $a, b \neq 0$, and $a \neq 1$. (d) Show that for each $A \in GL(2, \mathbb{C})$, there exists $B \in GL(2, \mathbb{C})$ such that $T_B^{-1} \circ T_A \circ T_B$ is in the form of one of the functions in the list above.