Non-Euclidean Geometry (spring 2011)

Partial Solutions to Exercise No. 4

1. Prove that if $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a Euclidean isometry with $f(0) = 0$, and C is a great circle on S^2, then $f(C)$ is a great circle.

2. Show that every isometry of S^2 is either the identity, reflection, rotation, or a rotation around a pair of points; followed by reflection in the great circle orthogonal to them.

   Solution: One way to solve this question is to use the fact that every isometry of S^2 is of the form $f(x) = Ax$ for $A \in O(3)$, and use question 3 from Daf Targil 3. Another way is to consider the fixed points of the map. If f fixes 3 spherically independent points then f is the identity. If f fixes 2 spherically independent points then f is a reflection. If f fixes 1 (spherically independent) point then f can be accomplished using two re reflections. But, as reflections in R^3, these are not parallel since both hyperplanes pass through the origin. So they give a rotation. The only remaining case is that f fixes no points on S^2 We know that $f(x) = Ax$ for some $A \in O(3)$. The eigenvalues of A are the roots of the equation $\det(\lambda I - A) = 0$, which is a cubic equation. Being a polynomial of odd degree, it has at least one real root, and hence at least one real eigenvalue μ. So $f(v) = Av = \mu v$ for some $v \in S^2$. Since $f(v) \in S^2, \|\mu v\| = 1$ and so $\mu = \pm 1$. But $\mu \neq 1$ since f has no fixed points. Thus $f(v) = -v$. Now consider the hyperplane $H = \{x \in \mathbb{R}^3 \mid v \cdot x = 0\}$. As an isometry of $\mathbb{R}^3$, f must map H to H (since f preserves angles). By changing coordinates, we can consider H to be a copy of $\mathbb{R}^2$. Then f restricted to H is an orientation preserving isometry of $\mathbb{R}^2$ that fixes the origin. So f restricted to H is a rotation about the origin. Thus f is a reflection through the great circle $C = H \cap S^2$, followed by a rotation about v.

3. Describe the isometry $g : S^2 \to S^2$ given by $g(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} (x)$. Hint: determine first how many fixed points g has.

   Solution: We see that $det(A) = -1$ where A is the corresponding matrix, so it is orientation reversing. Does it requires an odd number of reflections. Next we find the fixed points of g by solving $(A - I)v = 0$: the only solution is $x = y = z = 0$, but this isn't a point on S^2. So g fixes nothing (as an isometry of S^2). Thus, it isn't a reflection. From question 2 about we conclude that it must be a reflection combined with a rotation. We can find the axis of reflection by solving $A(v) = -v$ where $v = (x, y, z)$. The solutions are $k(1, -1, 1), k \in \mathbb{R}$, and since we only want solutions on S^2, we get the points $\pm(1, -1, 1)/\sqrt{3}$. Then g consists of a reflection through

$C = \{x \in S^2 \mid \langle v, x\rangle = 0\}$, followed by a rotation about v, where $v = \pm(1, -1, 1)/\sqrt{3}$. To find the rotation, choose a point u on C, say $u = (1, 1, 0)/\sqrt{2}$ (choose u such that $v \cdot u = 0$ and normalize). We can directly compute $u' = g(u) = (1, 0, -1)/\sqrt{2}$. The angle of rotation is exactly the angle between u and u', so $\cos\theta = u \cdot u' = 1/2$. Thus the rotation is by $\pm\pi/3$. We have still only specified both v and u up to sign. The two choices determine each other. Let us choose $\theta = \pi/3$. Then we would have $u \times u' = v \sin\theta$, $(-1, 1, -1)/2 = v\sqrt{3}/2$, and $(-1, 1, -1)/sqrt3 = v$. So g consists of a rotation of $+\pi/3$ around $v = (-1, 1, -1)/\sqrt{3}$, followed by a reflection in the great circle orthogonal to v. Had we chosen $\theta = -\pi/3$, we would have obtained $v = (1, -1, 1)/\sqrt{3}$.

4. Let $v \in S^2$, a great circle C orthogonal to v, and an angle θ. Set f to be the rotation by θ around v, and g a reflection in C. Show that $fg = gf$.

   Solution: The great circle C is given by $C = \{u \in S^2 \mid u \cdot v = 0\}$. Take two points b and c on C such that $b \neq \pm c$. Then, the points v, b and c do not lie on a common great circle and hence spherically independent. Let $b' = f(b), c' = f(c)$. Note that b', c' are both on C. The points b, b', c, c' are all on C and so are fixed by g. The points $\pm v$ are both fixed by f. Now:

   $$g(f(v)) = g(v) = -v,\ g(f(b)) = g(b') = b',\ g(f(c)) = g(c') = c'$$

   $$f(g(v)) = f(-v) = -v, f(g(b)) = f(b) = b',\ f(g(c)) = f(c) = c'$$

   Recall that an isometry is determined by its action on a (spherical) basis. Since fg and gf agree on the (spherical) basis v, b, c they must be equal.

   Another solution would be the following: write down a generic formula for the rotation and re flection. The rotation will need a change of basis matrix: the first two columns of this matrix will be orthogonal to v, the third will be v. Based on this orthogonality, we can argue that the matrix product is commutative

5. (*) Find all spherical triangles with angles $(\alpha, \beta, \gamma) = (\pi/p, \pi/q, \pi/r)$ where p, q, r are positive natural numbers. In each case deduce the number of triangles necessary to tile the sphere and calculate $V - F + E$ for the resulting tessellation.

   Solution (sketch): It follows from the formula for the area of a spherical triangle that any such triangle $(\pi/p, \pi/q, \pi/r)$, where N copies of which cover the sphere, must satisfy the Diophantine equation $N/p + N/q + N/r = N + 4$. The solutions of this Diophantine equation are: $(2, 3, 3), (2, 3, 4), (2, 3, 5), (1, n, n), (2, 2, n)$ for $n = 2, 3, \ldots$. Four examples are shown in a pdf file in the course webpage.

6. Let $a, b, c, d \in \mathbb{C}$ with $ac - bd \neq 0$, and define a map $T : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ by $T : z \to \frac{az+b}{cz+d}$. Show that these maps (called Möbius transformations) form a group (denoted below by Möb) under composition.

   Solution: Let $f_1(z) = \frac{a_1z+b_1}{c_1z+d_1}$ and $f_2(z) = \frac{a_2z+b_2}{c_2z+d_2}$. Then

   $$f_1(f_2(z)) = \frac{a_1\frac{a_2z+b_2}{c_2z+d_2} + b_1}{c_1\frac{a_2z+b_2}{c_2z+d_2} + d_1} = \frac{a_1a_2z + a_1b_2 + b_1c_2z + b_1d_2}{c_1a_2z + b_2c_1 + d_1c_2z + d_1d_2}$$

Moreover, it is easy to check that $h(z) = \frac{1z+0}{0z+1}$ is the identity, and that $g(z) = \frac{dz-b}{-cz+a}$ is the inverse of the Möbius transformation $f(z) = \frac{az+b}{cz+d}$.

7. Show that the map $\phi : SL(2,\mathbb{C}) \to$ Möb, given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \frac{az+b}{cz+d}$ is a group homomorphism and find its kernel. The group $SL(2,\mathbb{C})/\text{Ker}(\phi)$ (which is isomorphic to Möb) is called the projective special linear group.

8. Show that the Möbius transformation $z \mapsto z+1$ is conjugate to its inverse in Möb .

   Hint: Note that $f(z) = z+1$ is represented by $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and it's inverse $g(z) = z-1$ is represented by $B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. By a direct computation one can find $D \in$ Möb such that $DAD^{-1} = B$.

9. Let $a,b,c,d \in \mathbb{R}$, let $z \in \mathbb{C}$, and define $f(z) = \frac{az+b}{cz+d}$. Show that if $ad-bc \leq 0$ then $f(z)$ is not a one-to-one (or onto) map from the upper half plane $\mathbb{H}^2$ to itself, and that if $ad-bc > 0$ then $f(z)$ is a one-to-one and onto map from $\mathbb{H}^2 \to \mathbb{H}^2$.

   Solution: It is not hard to check that the imaginary part of $\frac{az+b}{cz+d}$ is $\frac{(ad-bc)\text{Im}(z)}{|cz+d|^2}$ in general, and so when $ad-bc=0$, we have an imaginary part of 0 and therefore there will be points in the upper half plane that get mapped to real numbers. Likewise if $ad-bc < 0$ then there will be points in the upper half plane that get mapped to points in the lower half plane. Thus, it is not a map from $\mathbb{H}^2 \to \mathbb{H}^2$. There is nothing to show for one-to-one or onto since it isn't even a function from $\mathbb{H}^2$ to itself. Next, if $ad-bc > 0$ then the function $f(z)$ is indeed a one-to-one and onto map from $\mathbb{H}^2 \to \mathbb{H}^2$. First we show $f(z) = \frac{az+b}{cz+d}$ is one-to-one. Consider two points z and w and suppose $f(z) = f(w)$. Then $\frac{az+b}{cz+d} = \frac{aw+b}{cw+d}$. By cross multiply one has $aczw + adz + bcw + bd = aczw + bcz + adw + bd$. Thus $(ad-bc)(z) = (ad-bc)(w)$. Since $ad-bc$ is positive by assumption here, we can cancel this term and get $z = w$. To show that $f(z)$ is onto, notice that if $ad-bc \neq 0$, then $f^{-1}(w) = \frac{dw-b}{-cw+a}$ is the inverse to $f(z)$ and so $f^{-1}(w) = z$ will be the point that maps to w under f. Note that w is a complex number in the upper half plane so cannot be zero. Secondly, if $a = c = 0$ then $ad-bc = 0$ which is a contradiction. So either a or wc is nonzero. Also, a cannot equal wc since one is real and the other imaginary.

10. (a) Show that any Möbius transformation, other than the identity map, has either one or two fixed points. (b) Let $A \in GL(2,\mathbb{C})$, and let $(z_1,z_2)^T$ be an eigenvector for the matrix A. Show that z_1/z_2 is a fixed point for the Möbius transformation $T_A(z) = \frac{az+b}{cz+d}$. (c) Find the fixed points of $h_b(z) = z+b, k_a(z) = az$, and $i(z) = 1/z$, where $a,b \neq 0$, and $a \neq 1$. (d) Show that for each $A \in GL(2,\mathbb{C})$, there exists $B \in GL(2,\mathbb{C})$ such that $T_B^{-1} \circ T_A \circ T_B$ is in the form of one of the functions in the list above.