Non-Euclidean Geometry (spring 2011)

Exercise No. 5 - Möbius transformations

1. Prove that two points $w, z \in \overline{\mathbb{C}}$ correspond to antipodal points in S^2 under stereographic projection if, and only if, $w = J(z)$ for the transformation $J(z) = -1/\overline{z}$. Show that any Möbius transformation T other than the identity has either one or two fixed points on $\mathbb{C} \cup \{\infty\}$. Show that the Möbius transformation corresponding under stereographic projection to a non-trivial rotation has two antipodal fixed points. Show that a Möbius transformation $T : z \to (az + b)/(cz + d)$, with $ad - bc = 1$ satisfies $J^{-1}TJ = T$ precisely when $d = \overline{a}$ and $c = -\overline{b}$.

2. Let g, h be two Möbius transformations with real coefficients (i.e., the corresponding matrices lie in $SL(2, \mathbb{R})$, so that g is parabolic. Assume that $g(y) = y$ and $h(y) \neq y$ for some $y \in \overline{\mathbb{C}}$. Does the commutator $f = ghg^{-1}h^{-1}$ parabolic? hyperbolic? elliptic?

3. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a transformation preserving the cross-ratio

$$D(x, y, z, w) = D(f(x), f(y), f(z), f(w))$$

for all pair-wise distinct points x, y, z, w. Show that f is a Möbius transformation. Hint: prove first that f is ono-to-one and onto and then look at the map $g(z) = D(a, b, c, z)$, for some fixed points a, b, c.

4. Let S be a circle in $\overline{C}$, and let f be a Möbius transformation. Let h be the refection over S. Prove that fhf^{-1} is the refection over $f(S)$. (Note that the statement is trivial when in addition f is a Euclidean isometry).

5. Show that every refection of $\overline{\mathbb{C}}$ is of the form $f(z) = \frac{a\overline{z}+b}{c\overline{z}+d}$, where $a, b, c, d \in \mathbb{C}$, and $ad - bc = 1$. Show that the opposite statement is false (that is give an example of a transformation of the above form which is not a reflection.)

6. Let $f \neq 1$ be a Möbius transformation. Show that the cross-ratio $D(z, fz, f^2z, f^3z)$ does not depend on the choice of the point z (whenever it is defined). Express this quantity in terms of $tr^2(f)$, and explore the cases when f is of order 2 and 3.

7. Find all of the Möbius transformations that commute with $z \mapsto kz$ for a fixed k.

8. Show that inversion maps any circle to another circle. Show that inversion preserves the magnitude of angles but reverses their orientation.

9. Show that the composition of an even number of inversions is a Möbius transformation.

10. Show that any loxodromic transformation is the composite of 4 inversions.