Non-Euclidean Geometry (spring 2011)

Partial solutions to exercise No. 5 - Möbius transformations

1. Prove that two points $w, z \in \overline{\mathbb{C}}$ correspond to antipodal points in S^2 under stereographic projection if, and only if, $w = J(z)$ for the transformation $J(z) = -1/\overline{z}$. Show that any Möbius transformation T other than the identity has either one or two fixed points on $\mathbb{C} \cup \{\infty\}$. Show that the Möbius transformation corresponding under stereographic projection to a non-trivial rotation has two antipodal fixed points. Show that a Möbius transformation $T : z \to (az + b)/(cz + d)$, with $ad - bc = 1$ satisfies $J^{-1}TJ = T$ precisely when $d = \overline{a}$ and $c = -\overline{b}$.

2. Let g, h be two Möbius transformations with real coefficients (i.e., the corresponding matrices lie in $SL(2, \mathbb{R})$, so that g is parabolic. Assume that $g(y) = y$ and $h(y) \neq y$ for some $y \in \overline{\mathbb{C}}$. Does the commutator $f = ghg^{-1}h^{-1}$ parabolic? hyperbolic? elliptic?

   Solution: Since g is parabolic, it is conjugate to $z \mapsto z + 1$ (we moved the point y to infinity). Now let $h(z) = \frac{az+b}{cz+d}$. Since ∞ is not a fixed point of h it follows that $c \neq 0$. A direct computation shows that $\text{trace}(f) = 2 + c^2$ and hence $\text{trace}(f) > 2$ which implies that f is hyperbolic.

3. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a transformation preserving the cross-ratio

   $$D(x, y, z, w) = D(f(x), f(y), f(z), f(w))$$

   for all pair-wise distinct points x, y, z, w. Show that f is a Möbius transformation. Hint: prove first that f is ono-to-one and onto and then look at the map $g(z) = D(a, b, c, z)$, for some fixed points a, b, c.

   Solution: Note first that f is one-to-one (otherwise, if $f(a) = f(b)$ for $a \neq b$, we can find points c and d such that the right-hand-side of the equality would be zero while the left-hand-side won't), and onto (otherwise the given equality would not hold). Next, let $g(z) = D(a, b, c, z)$. It is easy to check that g is Möbius. Since f is invertible, the map $g(f^{-1}(z) = D(a, b, c, f^{-1}(z)) = D(f(a), f(b), f(c), z)$ is also Möbius (since $H(z) = D(f(a), f(b), f(c), z)$ is). From this we conclude that f must be a Möbius transformation as well.

4. Let S be a circle in $\overline{C}$, and let f be a Möbius transformation. Let h be the refection over S. Prove that fhf^{-1} is the refection over $f(S)$. (Note that the statement is trivial when in addition f is a Euclidean isometry).

   Solution: Let $g(z) = f \circ h \circ f^{-1}(z)$, and denote by D^-, D^+ the "interior" of $f(S)$" and the "exterier" of $f(S)$ accordingly. Note that (i) $g(f(S)) = f(S)$, (ii) $g^2 = \text{Id}$, (*iii*) $g(D^+) = D^-$ and $g(D^-) = g(D^+)$ (indeed, this follows from the continuity of the map and the fact that h is a relection over S.), (*iv*) g preserves angle and the cross-ratio (since f,h, and f^{-1} preserve it). Combine this with what we proved in class we obtain that fhf^{-1} is the refection over $f(S)$

5. Show that every refection of $\overline{\mathbb{C}}$ is of the form $f(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$, where $a, b, c, d \in \mathbb{C}$, and $ad - bc = 1$. Show that the opposite statement is false (that is give an example of a transformation of the above form which is not a reflection.)

   Solution: First, assume that f is a reflection w.r.t a (genuine) line l. It is not hard to check that after applying a transformation of the form $g(z) = \alpha z + \beta$, where $|\alpha| = 1$, we can "transform" l to the real-axis. From the previous question it follows that gfg^{-1} is a reflection of the x-axis since $g(l) = \mathbb{R}$. In other words, $gfg^{-1}(z) = \bar{z}$, and hence $f(z) = \frac{\bar{\alpha}\bar{z}+\bar{\beta}-\beta}{\alpha}$. Moreover, note that since $|\alpha| = 1$, the condition "$ad - bc = 1$" for the map f is satisfied. Next, assume that f is a reflection w.r.t a (genuine) circle S. We first "translate" S to the unit circle using the map $T(z) = (z - v)/R$, where R is the radius, and v is the center of the circle S. Again, using the previous question, one has that $T^{-1}fT$ is a reflection w.r.t the unit circle S^1, i.e, $T(z) = -1/\bar{z}$. From this we can conclude by direct computation that $f(z) = \frac{v\bar{z}+(R^2-|v|^2)}{\bar{z}-\bar{v}}$. After resealing (both the numerator and a denominator) by iR, the claim follows.

   To see that the opposite statement is false, one can consider the map $f(z) = -4\bar{z}$ (by looking at the fixed points, it's clear that f is not a reflection).

6. Let $f \neq 1$ be a Möbius transformation. Show that the cross-ratio $D(z, fz, f^2z, f^3z)$ does not depend on the choice of the point z (whenever it is defined). Express this quantity in terms of $tr^2(f)$, and explore the cases when f is of order 2 and 3.

7. Find all of the Möbius transformations that commute with $z \mapsto kz$ for a fixed k.

   Solution: It is easy to check that the only Möbius transformations that commute with $z \mapsto kz$ are of the form $z \mapsto k'z$.

8. Show that inversion maps any circle to another circle. Show that inversion preserves the magnitude of angles but reverses their orientation.

   Solution: Consider a circle γ, with centre O and radius r. Recall that an inversion in γ is the transformation that sends a point X other than the origin to the point X' on the line OX, on the same side of O as X, such that $|OX'| \cdot |OX| = r^2$. Inversion clearly sends straight lines through O to themselves. It sends straight lines not through O to circles through O let l be a straight line not through O. Drop the perpendicular from O to l, and let the foot of this perpendicular be P. Let its image under the inversion be P'. Let Q be another point on l, and let Q' be its image. We know that $|OP| \cdot |OP'| = |OQ| \cdot |OQ'| = r^2$, so P, P', Q, and Q' are concyclic. Since $\angle OPQ = \pi/2$, we get that $\angle OQ'P' = \pi/2$. Therefore, Q' lies on the circle with diameter OP'. This circle is therefore the image of l. (We think of O as the image of ∞ under the inversion.) Finally, given a circle S that does not pass through O, the inversion of S in γ is another circle. Let l_1 be the line from O to the centre of S, and let l_2 be another line through O that meets S. Let P_1 and P_2 be the points where l_1 meets S, with P_1 nearer to O, and let Q_1 and Q_2 be the points where l_2 meets S, with Q_1 nearer to O. Let the images of these points be P_1', P_2', Q_1' and Q_2' respectively. Since $|OP_1| \cdot |OP_1'| = |OP_2| \cdot |OP_2'| = r^2$, we get that $\frac{|OP_2'|}{|OP_1|} = \frac{|OP_1'|}{|OP_2|} = \frac{r^2}{|OP_1| \cdot |OP_2|}$. This means that the image of S is an enlargement of S about O. This sends circles to circles, so the image of S is a circle.

Recall that $f : \mathbb{C} \to \mathbb{C}$ is said to be orientation reversing or anticonformal if f preserves the cosine of the angle between two intersecting curves and reverses the orientation. A classical fact from complex analysis gives that f is orientation reversing if and only if $\bar{f}$ is holomorphic. Combine this with the fact that any inversion is of the form $z \mapsto \frac{r^2}{\bar{z} - \bar{a}} + a$ completes the proof.

Another way to see that an inversion preserves the magnitude of an angle is the following: Recall from calculus that the angle between two circles $k_i(x^2 + y^2) + a_i x + b_i y + c_i = 0$, for $i = 1, 2$ at a common point, (x_0, y_0) is the angle between their (not necessarily unit) normals at that point, which are: $(2k_1x_0 + a_1, 2k_1y_0 + b_1)$, and $(2k_2x_0 + a_2, 2k_2y_0 + b_2)$. Hence the cosine of the angle between them is

$$\cos = \frac{a_1a_2 + b_1b_2 - 2k_1c_2 - 2k_2c_1}{\sqrt{a_1^2 + b_1^2 - 4k_1c_1}\sqrt{a_2^2 + b_2^2 - 4k_2c_2}}$$

Finally, check that if $k(x^2 + y^2) + ax + by + c = 0$ is the equation of the circle, then the equation of its inverse is $c(x^2 + y^2) + ax + by + k = 0$, and note that in the above expression for the cosine of the angle between two circles, it does not change if you swap k and c. This proves that inversion preserves the magnitude of angles.

9. Show that the composition of an even number of inversions is a Möbius transformation.

   Solution: Let C be the particular inversion $z \to \bar{z}$. If Γ is a circle, we can find a Möbius transformation T that maps $\mathbb{R} \cup \infty$ onto Γ. Then inversion in Γ is given by $J = T \circ C \circ T^{-1}$. If $T(z) = \frac{az+b}{cz+d}$, then $C \circ T \circ C(z) = \frac{\bar{a}z+\bar{b}}{\bar{c}z+\bar{d}}$. This shows that $C \circ T \circ C$ is a Möbius transformation. Hence

   $$C \circ J = C \circ T \circ C \circ T^{-1} = (C \circ T \circ C) \circ T^{-1}$$

   is a Möbius transformation, as is its inverse $J \circ C$. Now if J_1, J_2 are two inversions we can write $J_2 \circ J_1 = (J_2 \circ C) \circ (C \circ J_1)$ to show that $J_2 \circ J_1$ is a Möbius transformation. In particular, the composition of an even number of inversions is a Möbius transformation.

10. Show that any loxodromic transformation is the composite of 4 inversions.