Non-Euclidean Geometry (spring 2011)

Exercise No. 6 - Hyperbolic Geometry

1. Use what we did in class to prove that for any open ball $B \subset \overline{\mathbb{C}}$ and $z_1, z_2 \in B$, there is a unique circle $D \subset \overline{\mathbb{C}}$ passing through z_1 and z_2 such that $D \perp \partial B$.

2. Let $\rho_\triangle, \rho_\mathbb{H}$ be the hyperbolic distances in the unit disc model $\triangle = \{|z| < 1\}$, and in the upper-half plane model $\mathbb{H} = \{\mathrm{Im} z > 0\}$ respectively. Show that

$$\cosh \rho_\triangle(z_1, z_2) = \frac{2|z_1 - z_2|^2}{(1 - |z_1|^2)(1 - |z_2|^2)} + 1, \quad \text{for all } z_1, z_2 \in \triangle,$$

$$\sinh\Big(\frac{1}{2}\rho_\mathbb{H}(z_1, z_2)\Big) = \frac{|z_1 - z_2|}{2\sqrt{\mathrm{Im} z_1 \cdot \mathrm{Im} z_2}}, \quad \text{for all } z_1, z_2 \in \mathbb{H}$$

3. Find the set of points in $\triangle$ which are equidistant from points 0 and 1/3 w.r.t. $\rho_\triangle$.

4. In the upper-half plane model, a hyperbolic circle of radius r with center z is the set $S(z, r) = \{w \in \mathbb{H} : \rho_\mathbb{H}(z, w) = r\}$. Show that $S(z, r)$ is necessarily a Euclidean circle.

5. Complete the details (if necessary) in the proofs we gave in class of the following: Let $\triangle$ be a hyperbolic triangle with angles α, β, γ and corresponding side lengths a, b, c.

   (i) The hyperbolic law of sines:

$$\frac{\sinh a}{\sin \alpha} = \frac{\sinh b}{\sin \beta} = \frac{\sinh c}{\sin \gamma}$$

   (ii) The second hyperbolic law of cosines:

$$\cos(\gamma) = -\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\cosh(c)$$

   Note that (ii) implies that the angles of a triangle determines its side lengths.

6. Show that there is no local isometry between the Euclidean and the hyperbolic planes.

7. Show that the bisectors of a hyperbolic triangle meet at a common point.

8. Show that the angle sum of a hyperbolic triangle is strictly less than π.

9. Show that in hyperbolic space there exists a unique perpendicular from a point P to a line L. This perpendicular minimizes the distance from P to L.

10. (*) Let $L, L' \subset \mathbb{H}$ be disjoint lines which do not meet at ∞. Show that in contrast to Euclidean geometry, L, L' have a **unique** common perpendicular which minimizes the distance between them. Moreover, (also in contrasts with Euclidean space), perpendicular projection onto a line L strictly decrease distances.